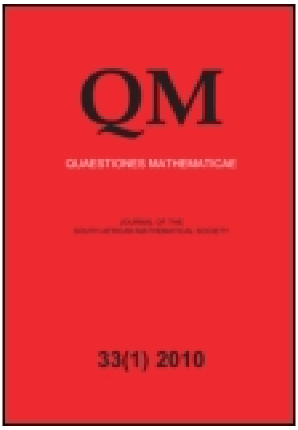

Quaestiones Mathematicae
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tqma20

REMARKS ON COMPLETELY CONTINUOUS POLYNOMIALS

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Version of record first published: 17 Sep 2010.

To cite this article: F. Bombal & G. Emmanuele (1997): REMARKS ON COMPLETELY CONTINUOUS POLYNOMIALS, Quaestiones Mathematicae, 20:1, 85-93

To link to this article: http://dx.doi.org/10.1080/16073606.1997.9631856

Quaestiones Mathematicae
20(1997), 85–93

REMARKS ON COMPLETELY CONTINUOUS POLYNOMIALS

F. BOMBAL* AND G. EMMANUELE

ABSTRACT.
We present some results pointing out pairs E, F of Banach spaces for which any polynomial $P : E \to F$ is completely continuous. Hence we study local complete continuity of holomorphic functions.

1991 *Mathematics Subject Classification.* 46B20, 46B28, 46G20.

1. Introduction. It is well known that several many isomorphic properties of Banach spaces are defined in terms of the coincidence of different classes of linear, bounded operators (see, for instance, the survey paper [B2]). Working in this direction many authors have studied the question of when two classes of polynomials between Banach spaces coincide. One of the first results of this kind is contained in old papers by Pelczynski ([P1], [P2]) where he considered assumptions on E, F sufficient to guarantee that any polynomial from E into F is completely continuous. Other more recent papers concerning the question of the validity of the equality $P(^kE, F) = P_{cc}(^kE, F)$ are the papers [GJ],[GG1],[GG2]. In this note we present some other results about this question.

Since polynomials enter quite naturally in the definition of holomorphic functions on infinite dimensional Banach spaces, i.e. functions possessing a Taylor expansions in a series of polynomials, the question of when a property enjoyed by polynomials passes to holomorphic functions

The present work was performed during a three month visit (April - June 1994) of the second author to Madrid thanks to financial support from Universidad Complutense.

*Partially supported by DGCYT grant PB94-0242 .

has been considered, too; for instance, it is known that compactness or weak compactness of polynomials in the Taylor series is equivalent to compactness or weak compactness of the corresponding holomorphic function (see [AS], Prop. 34; [R], Th. 3.2). Here, we present a result about the (local) complete continuity of holomorphic functions.

2. Preliminaries. Let E, F be two Banach spaces. For $k \geq 1, k \in N$ we shall denote by $\mathcal{L}(^k E, F)$ the space of all continuous k-linear operators from $E^k = E \times \overset{k}{\ldots} \times E$ into F. There is a canonical isomorphism between $\mathcal{L}(^k E, F)$ and $\mathcal{L}(\widehat{\bigotimes}^k E, F)$ where $\widehat{\bigotimes}^k E$ denotes the k-fold projective tensor product. A map $P : E \to F$ is a k-homogeneous polynomial if it is the restriction to the diagonal of E^k of a continuous k-linear map T, which is unique if it is required to be symmetric. $P(^k E, F)$ will denote the Banach space of all k-homogeneous polynomials from E into F, endowed with the sup norm on the unit ball of E. The closed linear span of the set $\{x^{(k)} = x \otimes \overset{k}{\ldots} \otimes : x \in E\}$ in $\widehat{\bigotimes}^k E$ is denoted by $\widehat{\Delta}^k E$. For every Banach space F, the spaces $\mathcal{L}(\widehat{\Delta}^k E, F)$ and $P(^k E, F)$ are isomorphic. If T_p is the operator corresponding to P and $\Theta_k : E \to \widehat{\Delta}^k E$ stands for the map $\Theta_k(x) = x^{(k)}$, then we have $P = T_P \circ \Theta_k$. In the note we shall also consider $P_{cc}(^k E, F)$, i.e. the subspace of $P(^k E, F)$ of all polynomials (called *completely continuous*) sending weakly converging sequences into norm converging sequences.

We also need some notions related to holomorphy in Banach spaces. Let E, F be complex Banach spaces and let U be an open subset of E. Recall that a map $f : U \to F$ is *holomorphic* if for every $x \in U$ there is a sequence of homogeneous polynomials $P_n f(x) \in P(^n E; F)$ and $r > 0$ such that $f(z) = \sum_{n=0}^{\infty} P_n f(x)(z - x)$, uniformly for $\|z - x\| \leq r$. The supremum of all such $r > 0$ is called the *radius of uniform convergence* of f at x, and will be denoted by $r(f, x)$. By the Cauchy-Hadamard formula ([M]), we have $[r(f,x)]^{-1} = \limsup \|P_n f(x)\|^{\frac{1}{n}}$. We shall denote by $H(U; F)$ the space of all the holomorphic mappings from U into F.

3. On completely continuous polynomials. This section is devoted to the question of when each polynomial between two Banach spaces is completely continuous. In particular, we point out two pairs of families of Banach spaces for which the equality $P(^k E, F) = P_{cc}(^k E, F)$ is true.

Let us first establish the following general stability result:

PROPOSITION 1. *Let E, F be a pair of Banach spaces such that*

$P(^kE,F) = P_{cc}(^kE,F)$. Then the same equality holds for the pair H, M, when:
1) H is a complemented subspace of E and M is a (closed) subspace of F;
2) $H = E/G$, with G a closed subspace not containing copies of ℓ_1, M as in (1).

Proof. (1) is obvious. As for (2), let $P \in P(^kE/G,M)$ and (q_n) be a weakly convergent sequence in E/G. By a result in [L], there is a weakly Cauchy sequence (x_n) in E such that $\pi(x_n) = q_n$, for every n (π denotes the quotient map from E onto E/G). Then, $Q = P \circ \pi \in P(^kE,M) = P_{cc}(^kE,M)$ by (1). By [AHV, Th.2.3], Q transforms the weakly Cauchy sequence (x_n) into a norm convergent sequence $Q(x_n)P(q_n)$. Hence P is completely continous.

In particular, if $P(^kE,F) = P_{cc}(^kE,F)$ for some non trivial F, then $P(^kE,\mathbf{K}) = P_{cc}(^kE,\mathbf{K})$, i.e., every scalar k-homogeneous polynomial on E is weakly sequentially continuous. This is equivalent to say (see section 2) that Θ_k ***sends weakly convergent sequences into weakly convergent sequences*** (in short, E has the k-SCP or ***k-sequential continuity property***). If, moreover, F is a Schur space, then clearly the k-SCP is a necessary and sufficient condition for E in order we have $P(^kE,F) = P_{cc}(^kE,F)$.

Examples of spaces with the k-SCP are the following:

a) If $1 \leq p$, the space ℓ_p has the k-SCP for $k < p$ and does not have the k-SCP for $k \geq p$ ([P1, 4.2]).

b) If the space $P_f(^kE)$ of finite type k-homogeneous polynomials (i.e., the space spanned by the monomials $(x^*)^k, x^* \in E^*$) is dense in $P(^kE)$, it is clear that E has the k-SCP.

c) The original Tsirelson's space T^* is a reflexive space with the k-SCP, for every $k \geq 1$ (see [AAD]).

d) Every Banach space E with the Dunford-Pettis property (see below for the definition) has the k-SCP for any $k \geq 1$ ([P2, Prop. 5]).

In particular, we have $P(^kT^*,\ell_1) = P_{cc}(^kT^*,\ell_1)$ for all $k \geq 1$ and $P(^k\ell_p,\ell_1) = P_{cc}(^k\ell_p,\ell_1)$ for $k < p \leq \infty$.

For our further results, we first need some preliminary facts, for which we need the following definition.

DEFINITION. A subset M of a Banach space E is named a Dunford-Pettis set (resp. limited set) if for any weak null (resp. weak* null) sequence $(x_n^*) \subset E^*$ we have

$$\lim_n \sup_{x \in M} x_n^*(x) = 0.$$

We need also the following result, which is interesting in itself.:

LEMMA 2. *Let $(x_n) \subset E$, $(y_n) \subset F$ be two Dunford-Pettis sequences, weakly converging to $x \in E$, $y \in F$, respectively. Then the sequence $(x_n \otimes y_n)$ is a Dunford-Pettis sequence in $E \otimes_\pi F$, that converges to $x \otimes y$ in the weak topology.*

Proof. First we show that $(x_n \otimes y_n)$ is a Dunford-Pettis sequence. Consider a weakly null sequence (L_n) in $(E \otimes_\pi F)^* = L(E, F^*)$. We have $L_n(x_n \otimes y_n) = [L_n(x_n)](y_n)$ for all $n \in N$. We claim that $L_n(x_n) \xrightarrow{w} \theta$ in F^*. Indeed, if $x^{**} \in F^{**}$ we have, for all $n \in N$, $[L_n(x_n)](x^{**}) = [L_n^*(x^{**})](x_n)$. Since the map $T \to T^*(x^{**})$ is linear and bounded from $L(E, F^*)$ into E^*, we have $L_n^*(x^{**}) \xrightarrow{w} \theta$ in E^*. Now we recall that (x_n) is a Dunford-Pettis sequence; so we may get our claim. Since, again, (y_n) is a Dunford-Pettis sequence, we have that $[L_n(x_n)](y_n) \to 0$, from which it easily follows that $(x_n \otimes y_n)$ is a Dunford-Pettis sequence. Now, we prove that $(x_n \otimes y_n) \xrightarrow{w} x \otimes y$. Choose $L \in L(E, F^*)$ and calculate as follows

$$\begin{gathered} L(x_n \otimes y_n) = \\ [L(x_n)](y_n) - [L(x)](y_n) + [L(x)](y_n) - [L(x)](y) + [L(x)](y) = \\ [L(x_n - x)](y_n) + L(x)[y_n - y] + L(x \otimes y). \end{gathered}$$

Observe now that $L(x_n - x) \xrightarrow{w} \theta$ and that (y_n) is a Dunford-Pettis sequence, so that the first summand of the last member of the previous equality goes to zero; since $y_n - y \xrightarrow{w} \theta$, also the second summand goes to zero. We are done.

COROLLARY 3. *Let (x_n) be a Dunford-Pettis sequence in E, weakly converging to $x \in E$. Then the sequence $(x_n \otimes x_n \otimes x_n \cdots x_n)$ is a Dunford-Pettis sequence in $E \otimes_\pi E \otimes_\pi \cdots E$, that converges to $x \otimes x \otimes \cdots x$ in the weak topology.*

To show this result it is enough to apply an inductive argument and Lemma 2. Similarly, we can show the following.

LEMMA 4. *Let $(x_n) \subset E$, $(y_n) \subset F$ be two limited sequences, weakly converging to $x \in E$, $y \in F$, respectively. Then the sequence $(x_n \otimes y_n)$ is a limited sequence in $E \otimes_\pi F$, that converges to $x \otimes y$ in the weak topology.*

COROLLARY 5. *Let (x_n) be a limited sequence in E, weakly converging to $x \in E$. Then the sequence $(x_n \otimes x_n \otimes x_n \cdots x_n)$ is a limited sequence in $E \otimes_\pi E \otimes_\pi \cdots E$, that converges to $x \otimes x \otimes \cdots x$ in the weak topology.*

We also need to consider the following properties of Banach spaces.

DEFINITION. A Banach space E is said to possess the Dunford-Pettis property (resp. Gelfand-Phillips property) if any relatively weakly compact (resp. limited) set in E is a Dunford-Pettis (resp. relatively compact) set.

Many examples of spaces possessing the Dunford-Pettis property or the Gelfand-Phillips property are known in the literature (we refer to [D],[DU],[E2] for lists of such spaces).

We are now able to prove the main result of this section.

THEOREM 6. *Suppose E and F are two Banach spaces verifying at least one of the following two conditions:*
1) E has the Dunford-Pettis property and in F any Dunford-Pettis set is relatively compact.
2) In E any weak null sequence is limited and F has the Gelfand-Phillips property.
Then $P(^kE, F) = P_{cc}(^kE, F)$, for all $k \in N$.

Proof. Let (x_n) be a sequence in E weakly converging to some $x \in E$. Under (1) it is Dunford-Pettis, whereas under (2) it is limited. Since any $P \in P(^kE, F)$ admits a factorization as described in the previous section, thanks to our Lemmata 2 and 4 the sequence $(P(x_n))$ is Dunford-Pettis under (1) or limited under (2), and moreover it converges weakly to $P(x)$. Our assumptions on F allow us to conclude the proof.

Examples of spaces in which any Dunford-Pettis set is relatively compact are contained in the paper [E1], whereas the following are examples of spaces in which any weakly null sequence is limited:
1) Schur spaces (i.e. spaces in which weakly null sequences are strongly null).
2) Grothendieck spaces (see [D]) possessing the Dunford-Pettis property.

3) $\Lambda(X)$ with Λ a suitable sequence space and X verifying the required property (see [B1]).

Theorem 6 in conjunction with Proposition 1 and the subsequent commentaries, allow us to get a wide class of Banach spaces E, F such that $P(^kE,F) = P_{cc}(^kE,F)$. As already quoted, results similar to ours are contained in the recent papers [GG1], [GG2] by Gonzalez and Gutierrez, who actually have the following results:
R$_1$) If E has the Dunford-Pettis property, then $L(E,F) = WCo(E,F)$ (the weakly compact operators) implies $P(^kE,F) = P_{cc}(^kE,F)$ for all $k \in N$.
R$_2$) $L(E,c_0) = CC(E,c_0)$ (the completely continuous operators) implies $P(^kE,c_0) = P_{cc}(^kE,c_0)$ for all $k \in N$.
R$_3$) If E has the hereditary Dunford-Pettis property and $L(E,F) = UC(E,F)$ (the unconditionally converging operators), then $P(^kE,F) = P_{cc}(^kE,F)$ for all $k \in N$.

Assuming $E = F = l_1$ in Theorem 6, (1), or $E = l_1(l_\infty), F = l_1$ in Theorem 6, (2), gives that our result is not a consequence of R$_1$ or R$_2$. If we choose $E = l_\infty$ and F a Gelfand-Phillips space, then our Theorem 6 can be used, but R$_3$ is not applicable. Conversely, the choice $E = L_1$, F a reflexive Banach space proves that R$_1$ does not follow from Theorem 6, whereas if $E = c_0$ and F does not contain copies of c_0, R$_3$ is applicable, but Theorem 6 is not.

Concerning R$_2$ we observe that the assumption $L(E,c_0) = CC(E,c_0)$ is equivalent to saying that any weakly null sequence is limited; since $F = c_0$ is a Gelfand-Phillips space, the Gonzalez-Gutierrez result R$_2$ surely follows from Theorem 6, (2); actually, we have the following more precise result.

THEOREM 7. *Let F be a Gelfand-Phillips space containing c_0. Then the following are equivalent:*
i) $L(E,F) = CC(E,F)$;
ii) $L(E,c_0) = CC(E,c_0)$;
iii) $P(^kE,F) = P_{cc}(^kE,F)$;
iv) $P(^kE,c_0) = P_{cc}(^kE,c_0)$.

Proof. i) $\Rightarrow$ ii) and iii) $\Rightarrow$ iv) are clear since F contains c_0. ii) $\Leftrightarrow$ iv) is due to Gonzalez-Gutierrez ([GG2]). We prove that i) $\Rightarrow$ ii) $\Rightarrow$ iii); since F contains c_0, ii) follows from i) and so, under ii), any weakly null sequence in E is limited, as already remarked; being F a Gelfand-Phillips space, we have iii) by virtue of Theorem 6. Finally, we show that iii) $\Rightarrow$ i). Suppose $T \in L(E,F) \setminus CC(E,F)$. There is a weakly null sequence $(x_n) \subset E$ with $\|T(x_n)\| \not\to 0$. Hence $(T(x_n))$ is not limited, since F is a Gelfand-Phillips space; this means that there is a weak* null sequence $(y_n^*) \subset F^*$ for which $< T(x_n), y_n^* > \not\to 0$. Define a polynomial $H : E \to c_0$ by $H(x) = [(< T(x), y_n^* >)^k]$ for all $x \in E$. Since $c_0 \subset F$, $H \in P(^kE, F)$ and hence $H \in P_{cc}(^kE, F)$. This gives that $\|H(x_n)\|_{c_0} \to 0$, contradicting the choice of (y^*).

4. On locally completely continuous holomorphic functions.

Unlike to the case of (strong or weak) compactness quoted in section 2, it is not enough that all the polynomials in the Taylor expansion of a holomorphic function f are completely continuous to ensure that f is (i.e. that f sends weakly convergent sequences in U into norm convergent sequences in F), as the following example shows:

EXAMPLE. Let $E = c_0, F = \mathbf{C}$ and

$$f(x) = \sum_{n=1}^{\infty} x_n^n, \qquad \text{for } x = (x_n) \in c_0.$$

Then f is an entire function ([M, Ex. 5.5]) and, since c_0 has the Dunford-Pettis Property, every scalar polynomial on it is completely continuous. If we put $x_n = (-1)^n e_n$, then (x_n) is weakly null, but $f(x_n)$ does not converge.

Hence, additional assumptions are needed in this case. We recall that a characterization of the entire functions which sends weakly convergent sequences into norm convergent ones was given in ([A], Prop. 1.8).

Here we consider the notion of *local complete continuity* that will be seen the appropriate one to answer (at least partially) our question.

DEFINITION. A map $f \in H(U; F)$ is locally completely continuous at $x \in U$ if, whenever (x_n) converges weakly to x and $\|x_n - x\| \leq \rho < r(f,x)$ for every n, then $(f(x_n))$ is norm-convergent.

THEOREM 8. *Let $f \in H(U,F)$. The following assertions are equivalent:*
a) f is locally completely continuous at $x \in U$;
b) $P_n f(x)$ is completely continuous at 0, for every $n \in \mathbf{N}$.

Proof. a) $\Rightarrow$ b): Let f be locally completely continuous at $x \in U$ and fix $k \in \mathbf{N}$, $\epsilon > 0$ and let us write $P_k f(x) = P_k$ to simplify. Let (x_n) be a weakly null sequence. We have to prove that there exists $n_\epsilon \in \mathbf{N}$ such that $\|P_k(x_n)\| \leq \epsilon$ for $n \geq n_\epsilon$. Let $M > 0$ be suitably chosen in order that $\|Mx_n\| \leq \rho < r(f,x)$ for every n. Proceeding as in ([A], Prop. 1.8) we prove that there exists $n_\epsilon \in \mathbf{N}$ such that $\|f(y) - f(x)\| \leq M^k\epsilon$ for every $y \in x + \mathrm{aco}\{Mx_n : n \geq n_\epsilon\}$ (where aco(A) means the absolutely convex hull of the set A). If there is an $n > n_\epsilon$ such that $\|P_k(x_n)\| > \epsilon$, then we can find a $\phi \in F^*$ such that $|\phi(P_k(Mx_n))| > 1$ and $|\phi(z)| \leq 1$ for $\|z\| < M^k\epsilon$. If we define $g(\lambda) = \phi(f(x + \lambda M x_n) - f(x))$, then g is holomorphic in $\{\lambda \in \mathbf{C} : \lambda < \delta\}$, for some $\delta > 1$, and

$$|\phi(P_k(Mx_n))| = \left|\frac{g^{(k)}(0)}{k!}\right| > 1.$$

By Cauchy inequalities,

$$\left|\frac{g^{(k)}(0)}{k!}\right| \leq \sup_{|\lambda|=1} |g(\lambda)| = \sup_{|\lambda|=1} |\phi(f(x + \lambda M x_n) - f(x))| \leq 1,$$

which is a contradiction.

b) $\Rightarrow$ a): Let $(x_n) \xrightarrow{w} x$, $\|x_n - x\| \leq \rho < r(f,x)$. As above, let us write $P_k f(x) = P_k$. Then

$$\|f(x_n) - f(x)\| \leq \left\|\sum_{k=1}^{\infty} P_k(x_n - x)\right\| \leq \sum_{k=1}^{m} \|P_k(x_n - x)\| + \sum_{k>m} \|P_k\|\rho^k$$

Since $\sum_{k=1}^{\infty} \|P_k\|\rho^k < \infty$, given $\epsilon > 0$ we can choose $m \in \mathbf{N}$ such that

$$\sum_{k>m} \|P_k\|\rho^k < \frac{\epsilon}{2}.$$

Then, since each P_k is completely continuous at 0, we can find n_ϵ such that if $n \geq n_\epsilon$,

$$\sum_{k=1}^{m} \|P_k(x_n - x)\| < \frac{\epsilon}{2},$$

i.e.,

$$\|f(x_n - f(x)\| < \epsilon, \text{ for } n \geq n_\epsilon.$$

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Received 1 September 1995 .