PROPERTY (V) OF PELCZYNSKI IN PROJECTIVE TENSOR PRODUCTS

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(Communicated by S. Dineen, M.R.I.A.)

[Received 11 February 1994. Read 11 December 1995. Published 29 December 1995.]

ABSTRACT

We prove the following result: if two Banach spaces E and F have property (V) of Pelczynski and $W(E, F^*) = K(E, F^*)$, then $E \otimes_\pi F$ has the same property. Then we give several results on the necessity of the condition $W(E, F^*) = K(E, F^*)$.

In [10] Pelczynski introduced the following isomorphic property (which he called property (V), in symbols (PV)): *a Banach space E has (PV) if and only if any unconditionally converging operator $T : E \to F$, F an arbitrary Banach space, is weakly compact.*

It is known that $C(K)$ spaces have (PV) ([10]) as well as Banach spaces not containing l_1, but possessing property (u) (see [9] for this definition). Furthermore, if E does not contain l_1 and has property (u), then $C(K, E)$ has (PV) for any compact Hausdorff space K ([2]), whereas if K is dispersed then $C(K, E)$ has (PV) if and only if E has (PV) ([1]). Recently Godefroy and Saab ([6]) also proved that Banach spaces which are M-ideals in their bidual enjoy (PV). We also observe that (PV) is inherited by quotients and that duals of spaces with (PV) are weakly sequentially complete ([10]).

Pelczynski ([10]) gave the following useful characterisation of Banach spaces with (PV).

Theorem 1 ([10]). *A Banach space E has (PV) if and only if any (bounded) subset M of E^* such that*

$$\lim_n \sup_M | x_n(x^*) | = 0 \tag{1}$$

for any weakly unconditionally converging series $\sum x_n$ in E is relatively weakly compact.

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[1]Supported by M.U.R.S.T. of Italy (40%, 1991).

The purpose of this note is to present a result about (PV) in projective tensor products of Banach spaces, in this way introducing a new family of Banach spaces enjoying (PV).

In the statement of the main theorem we shall assume that each operator from E into $F^*(E, F$ the two Banach spaces involved in our construction) is compact. The second part of the note will show that this assumption is sometimes a necessary condition for $E \otimes_\pi F$ to possess (PV).

Throughout, by $L(E, F^*)$, $W(E, F^*)$, $K(E, F^*)$ we denote the Banach spaces of all linear and bounded, weakly compact and compact operators from E into F^* respectively.

First, we present the main theorem.

Theorem 2. *Let E, F be two Banach spaces with (PV) such that $W(E, F^*) = K(E, F^*)$. Then $E \otimes_\pi F$ has (PV).*

PROOF. First of all, we observe that $(E \otimes_\pi F)^* = L(E, F^*) = W(E, F^*)$ where the second equality holds because both E, F possess (PV). Hence our hypotheses imply that $(E \otimes_\pi F)^* = K(E, F^*)$. Now let us choose a subset M of $K(E, F^*)$ verifying (1). If we put $H = \overline{span}\{h_n(x) : x \in E,\ n \in N\}$, for an arbitrary sequence $(h_n) \subset M$, we get a separable closed subspace of F^*. Let $Y \subset F$ be a countable set separating H. If $y \in Y$, the sequence $(h_n^*(y)) \subset E^*$ has to verify (1). Indeed, let $\sum x_n$ be a weakly unconditionally converging series in E. The series $\sum x_n \otimes y$ is easily seen to be weakly unconditionally converging in $E \otimes_\pi F$, too. Hence we get

$$\lim_n h_n^*(y)(x_n) = \lim_n h_n(x_n \otimes y) = 0$$

because (h_n) verifies (1). Our claim follows: $(h_n^*(y))$ verifies (1). Property (V) of E implies that $(h_n^*(y))$ is a relatively weakly compact subset of E^*. We can assume (and we do) that $(h_n^*(y))$ is a weak Cauchy sequence in E^* for all $y \in Y$, thanks to the countability of Y (and by passing to a subsequence if necessary).

Now, let $x^{**} \in E^{**}$ and consider $(h_n^{**}(x^{**})) \subset F^*$. We prove that $(h_n^{**}(x^{**}))$ verifies (1). To this end, we consider a weakly unconditionally converging series $\sum y_n$ in F and we show that

$$\lim_n \langle h_n^{**}(x^{**}), y_n \rangle = \lim_n \langle x^{**}, h_n^*(y_n) \rangle = 0, \tag{2}$$

i.e. we show that $(h_n^*(y_n))$ is weakly null in E^*. To do that we shall prove that $(h_n^*(y_n))$ is weakly relatively compact and weak*-null. Let us consider another weakly unconditionally converging series $\sum x_n$ in E; we claim that $\sum x_n \otimes y_n$ is weakly unconditionally converging in $E \otimes_\pi F$; indeed, if $h \in L(E, F^*)$, we have that $\sum h(x_n)$ is weak unconditionally converging in F^*. Corollary 2 of [11] gives that $\sum | h(x_n \otimes y_n) | < \infty$, from which our claim follows. Hence, since (h_n) verifies (1), we have

$$\lim_n \langle h_n^*(y_n), x_n \rangle = \lim_n h_n(x_n \otimes y_n) = 0,$$

from which it follows that $(h_n^*(y_n))$ verifies (1) in E^*. This gives that $(h_n^*(y_n))$ is relatively weakly compact in E^*; on the other hand, for any $x \in E$, the series $\sum x \otimes y_n$ is weakly unconditionally converging in $E \otimes_\pi F$, and so

$$\lim_n \langle h_n^*(y_n), x\rangle = \lim_n h_n(x \otimes y_n) = 0,$$

from which it follows that $(h_n^*(y_n))$ is weak*-null in E^*. We can then conclude that $(h_n^*(y_n))$ is weak-null in E^*. Hence (2) follows; so $(h_n^{**}(x^{**}))$ verifies (1) and hence is a relatively weakly compact subset of F^*. Let z_1, z_2 be two weak sequential cluster points of $(h_n^{**}(x^{**}))$ that obviously belong to H, since each $h_n : E \to H$ is compact. If $y \in Y$ we have

$$\begin{aligned} z_1(y) &= \lim_n h_{k(n)}^{**}(x^{**})(y) = \lim_n x^{**}[h_{k(n)}^*(y)] \\ &= \lim_n x^{**}[h_n^*(y)] = \lim_n x^{**}[h_{p(n)}^*(y)] \\ &= \lim_n h_{p(n)}^{**}(x^{**})(y) = z_2(y) \end{aligned}$$

where $h_{k(n)}^{**}(x^{**}) \overset{w}{\to} z_1$, $h_{p(n)}^{**}(x^{**}) \overset{w}{\to} z_2$. Since Y was a set separating H, we get that $z_1 = z_2$. This means that for all $x^{**} \in E^{**}$ there is $\tilde{h}(x^{**}) \in H$ such that $\tilde{h}(x^{**}) = w - \lim_n h_n^{**}(x^{**})$. A result in [12] now implies that (h_n) is a weak Cauchy sequence in $K(E,H) \subset K(E,F^*)$ that is weakly sequentially complete, since E^*, F^* are ([12]). Hence (h_n) is weakly converging in $K(E,F^*)$. This concludes the proof. ■

Corollary 3. *Let E^*, F have (PV). If $W(E^*,F^*) = K(E^*,F^*)$, then the space $N_1(E,F)$ of all nuclear operators from E into F has (PV).*

PROOF. $N_1(E,F)$ is a quotient of $E^* \otimes_\pi F$. ■

Corollary 4. *Let $E = C(K)$, $F = l_p$, $p > 2$. Then $E \otimes_\pi F$ has (PV).*

Corollary 5. *Let E, F have (PV). If either E^* or F^* has Schur property, then $E \otimes_\pi F$ has (PV).*

Corollary 6. *Let E, $F \neq 0$ have the Dunford–Pettis property. Then the following are equivalent:*

(i) *either E or F does not contain copies of l_1 and E, F have (PV);*
(ii) *$E \otimes_\pi F$ has (PV).*

PROOF. If (i) is true, Corollary 5 gives (ii). If (ii) is true, it is enough to apply theorem 8 of [4] to get (i). ■

The next results are about the assumption $W(E,F^*) = K(E,F^*)$ considered in Theorem 2. They show that in several cases such an assumption is necessary for $E \otimes_\pi F$ to possess (PV).

Theorem 7. *Assume that one of the following hypotheses is verified:*

(a) *E^* or F^* has the metric approximation property;*

(b) *E or F has an unconditional compact expansion of the identity;*

(c) *E^* or F^* has the compact approximation property and is a subspace of a Banach space Z possessing an unconditional compact expansion of the identity.*

Then, if $E \otimes_\pi F$ has (PV), $W(E, F^*) = K(E, F^*)$.

PROOF. If $E \otimes_\pi F$ has (PV), then $L(E, F^*) = (E \otimes_\pi F)^*$ is weakly sequentially complete. Under (a), we have our thesis thanks to a result in [8]. If we assume that $W(E, F^*) \neq K(E, F^*)$ in the remaining cases (b) and (c), results contained in [7] and [5] respectively give the existence of a copy of c_0 inside $K(E, F^*)$ that is not allowed to be weakly sequentially complete. ■

Theorem 8. *Let F^* be complemented in a Banach space Z having an unconditional Schauder decomposition (Z_n) with $W(E, Z_n) = K(E, Z_n)$ for all $n \in N$. If $E \otimes_\pi F$ has (PV), then necessarily $W(E, F^*) = K(E, F^*)$.*

PROOF. Let us assume that $W(E, F^*) \neq K(E, F^*)$. Hence there is $T : E \to F^*$ that is not compact, whereas $P_n T$ is in $W(E, Z_n) = K(E, Z_n)$ for all $n \in N$ (here P_n denotes the projection of Z onto Z_n). Repeating the proof of the main theorem in [3], we construct a copy of c_0 inside $K(E, F^*)$, a contradiction that concludes the proof. ■

As a consequence of Theorem 8 we have that $C(K) \otimes_\pi l_p$, K not dispersed, $1 < p \leq 2$, and $l_p \otimes_\pi l_q$, $1 < p < q' < \infty$, q, q' dual numbers, do not enjoy (PV).

Theorem 9. *Let E have the Dunford–Pettis property. If F contains copies of l_1 and $E \otimes_\pi F$ has (PV), then $W(E, F^*) = K(E, F^*)$.*

PROOF. Using theorem 8 of [4] as in Corollary 6, we get that l_1 does not live in E. Hence E^* has Schur property. Our thesis follows. ■

Theorem 10. *Let E, F have the Dunford–Pettis property. If $E \otimes_\pi F$ has (PV), then $W(E, F^*) = K(E, F^*)$.*

PROOF. Corollary 6 implies that either E or F does not contain copies of l_1. Hence either E^* or F^* has Schur property. Our thesis follows. ■

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