# ON STRONGLY DUNFORD–PETTIS OPERATORS ON CERTAIN SPACES[1]

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ABSTRACT

A recent result by Holub states that if $E$ is a separable Banach space then any operator $T: L^1(m) \to E$ is a strongly Dunford–Pettis operator if and only if for any weak* null sequence $(x_n^*) \subset E^*$, the sequence $(T^*(x_n^*))$ converges in measure to $\theta$ in $L^\infty(m)$. In this short note we improve this result by assuming that $E$ is a Gel'fand–Phillips space.

In a recent paper [5] Holub introduced the notion of *strongly Dunford–Pettis operators* on $L^1(m)$, with $(S, F, m)$ a $\sigma$-finite measure space, trying to extend some well-known results about Dunford–Pettis operators on $L^1(m)$, with $(S, F, m)$ a finite measure space, to the case of infinite measure, because they are meaningless or false when passing from a finite to an infinite measure. In particular he characterised strongly Dunford–Pettis operators from $L^1(m)$ into *separable* Banach spaces $E$ by means of the behaviour of the conjugate operator $T^*$ on weak* null sequences of $E^*$. The purpose of this note is to improve this result by considering Gel'fand–Phillips spaces instead of merely separable ones; we also remark that our proof is simpler than Holub's.

**Definition 1.** An operator $T$ from $L^1(m)$ into $E$ is a Dunford–Pettis (resp. strongly Dunford–Pettis) operator if it maps relatively weakly compact (resp. uniformly integrable and bounded) subsets into relatively compact ones.

We refer to [1] and [5] for these definitions; we also recall that a subset $X$ of $L^1(m)$ is uniformly integrable if for every $\epsilon > 0$ there is $\delta > 0$ such that, if $A \in F$ with $m(A) < \delta$, then $\int_A |f(s)|\, dm < \epsilon$ for all $f \in X$ [3].

It is clear from Definition 1 and a well-known characterisation of relative weak compactness in $L^1(m)$ [3] that if the underlying measure space is finite, then strongly Dunford–Pettis operators are exactly Dunford–Pettis operators.

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**Definition 2** [2]. Let $X$ be a bounded subset of $E$. $X$ is called limited if for each weak* null sequence $(x_n^*)$ in $E^*$ one has

$$\lim_n \sup_X |x_n^*(x)| = 0.$$

$E$ is a Gel'fand–Phillips space if its limited subsets are relatively compact.

It is known that separably complemented Banach spaces, Banach spaces with weak* sequentially compact dual balls, Banach spaces with the Schur property and dual Banach spaces with the weak Radon–Nikodym property are Gel'fand–Phillips spaces. Hence the following theorem is strictly more general than Holub's result.

**Theorem.** *Assume that $E$ is a Gel'fand–Phillips space. Then any $T$ from $L^1(m)$ into $E$ is strongly Dunford–Pettis if and only if for any weak* null sequence $(x_n^*) \subset E^*$, the sequence $(T^*(x_n^*))$ converges in measure to $\theta$ in $L^\infty(m)$.*

Proof. One implication is in theorem 3.4 of [5]. We have only to show the sufficiency of the condition on $T^*$. Let $X$ be a uniformly integrable, bounded subset of $L^1(m)$. Since $E$ is a Gel'fand–Phillips space it suffices to show that $T(X)$ is a limited subset of $E$. Assume this is false. Then there is a weak* null sequence $(y_k^*)$ in $E^*$, a sequence $(f_k) \subset X$, and $H > 0$ such that

$$H < |(T(f_k))(y_k^*)| = |(T^*(y_k^*))(f_k)| \qquad k \in \mathbb{N}.$$

By hypothesis the sequence $(T^*(y_k^*))$ converges in measure to $\theta$ in $L^\infty(m)$; hence, by passing to a further subsequence if necessary, it converges almost uniformly to $\theta$ [3]. By the uniform integrability of $X$ given $H/3M$ ($M = \sup\|T^*(y_k^*)\|$) there is $\delta > 0$ such that for $A \in F$, $m(A) < \delta$, one has $\int_A |f_k(s)|\, dm < H/3M$, for all $k \in \mathbb{N}$. Now, consider an $A_\delta \in F$ with $m(A_\delta) < \delta$ and $(T^*(y_k^*))$ converging uniformly to $0$ outside of $A_\delta$. For $k$ sufficiently large we get

$$\sup_{S \setminus A_\delta} |T^*(y_k^*)(s)| < H/3B$$

where $B = \sup_k \|f_k\|$. For the same values of $k$ we have

$$\begin{aligned} H < |(T^*(y_k^*))(f_k)| &= \left| \int_S f_k(s)(T^*(y_k^*))(s)\, dm \right| \\ &\leqslant \left| \int_{A_\delta} f_k(s)(T^*(y_k^*))(s)\, dm \right| + \left| \int_{S \setminus A_\delta} f_k(s)(T^*(y_k^*))(s)\, dm \right| \\ &\leqslant M \int_{A_\delta} |f_k(s)|\, dm + B \sup_{S \setminus A_\delta} |T^*(y_k^*)(s)| < (2/3)H, \end{aligned}$$

a contradiction that finishes the proof.

**Corollary.** *Let $(S, F, m)$ be a finite measure space and $E$ be a Gel'fand–Phillips space. Then $T: L^1(m) \to E$ is a Dunford–Pettis operator if and only if $T^*$ maps weak* null sequences in $E^*$ into sequences in $L^\infty(m)$ converging in the $L^1(m)$-norm.*

This last result improves partially the following well-known fact: an operator from $L^1(m)$ into an arbitrary Banach space $E$ is Dunford–Pettis if and only if $T^*$ (unit ball of $E^*$) is relatively compact in $L^\infty(m)$ in the $L^1(m)$-norm [4, p. 65].

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