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CONVERGENCE OF THE MANN–ISHIKAWA ITERATIVE PROCESS FOR NONEXPANSIVE MAPPINGS

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SECTION 1

LET E be a Banach space and let X be a bounded, closed and convex subset of E. We consider a function f, $f: X \to X$, which is nonexpansive, i.e.

$$\|fx - fy\| \leq \|x - y\| \qquad \text{for each } x, y \in X.$$

In order to construct a fixed point of f, Ishikawa (see [2]) considered the following iterative process, which is a particular case of an iterative procedure introduced by Mann ([4]): let $x_1 \in X$ and let $\{t_n\}$ be a sequence of real numbers such that there is $b \in \mathbb{R}$ for which

$$0 \leq t_n \leq b < 1, \sum_{n=1}^{\infty} t_n = \infty.$$

The Mann–Ishikawa sequence is defined by

$$x_{n+1} = (1 - t_n)x_n + t_n f x_n \qquad \text{for all } n \in N.$$

In [2] there are proved two very important properties of the above defined sequence $\{x_n\}$; indeed, Ishikawa showed that

$$\lim_n \|x_n - fx_n\| = 0 \tag{1.1}$$

$$\|x_{n+1} - y\| \leq \|x_n - y\| \qquad \text{for all } n \in N \tag{1.2}$$

where, in (1.2), y is a fixed point of f, if it exists.

The purpose of this note (sections 2 and 3) is to show that under suitable assumptions on f, X, E we can guarantee that

(i) f has a fixed point
(ii) $\{x_n\}$ converges (either strongly or weakly) to a fixed point of f.

Moreover, in section 4, we prove briefly that certain conditions due to Petryshyn & Williamson (see [6]) can be extended to the Mann–Ishikawa sequences in order to show that (ii) is true; in this case, we suppose that $\mathcal{F}(f) \neq \emptyset (\mathcal{F}(f)$ the fixed point set of $f)$. Always in section 4, we prove that $\{x_n\}$ converges strongly to the unique fixed point of f, if f satisfies a suitable condition of contractive type with $\mathcal{F}(f) \neq \emptyset$.

At the end, in section 5, we prove that most of the Mann–Ishikawa iterative processes converge, in the sense that in a suitable complete metric space M of nonexpansive functions, there exists a dense G_δ-subset $\bar{\bar{M}}$ of M for which (i) and (ii) are true.

SECTION 2

In this section, we consider only conditions on f in order to prove our thesis.

(a_1) First of all, we consider a simple generalization of Theorem 1 of [2]. Indeed, we suppose that

$$\alpha(f(A)) < \alpha(A) \qquad A \subset X,\ \alpha(A) > 0$$

where $\alpha(Y)$, $Y \subset X$, is the Kuratowski measure of noncompactness of Y (see [1]).

It is easy to show that, by (1.1), it follows $\alpha(\{x_n\}) = 0$; and so, there are $\{x_{k(n)}\}$ and $y \in X$ for which $x_{k(n)} \xrightarrow{s} y$; by (1.1), $y = fy$ and by (1.2) a $d \geq 0$ exists for which $\lim_n \|x_n - y\| = d$; since $\lim_n \|x_{k(n)} - y\| = 0$, we obtain $d = 0$. Then, $x_n \xrightarrow{s} y$.

(a_2) Now, we suppose that there is g, g: $\mathbb{R}^+ \to \mathbb{R}$, with $g(r) < r$ if $r > 0$, g continuous from the right and nondecreasing, for which, for each $x, y \in X$,

$$(fx - fy, j)_+ = (fx - fy, x - y)_+ \leq g(\|x - y\|)\, \|x - y\| \tag{2.1}$$

(or

$$(fx - fy, j)_+ = (fx - fy, x - y)_+ \leq g(\|x - y\|^2)) \tag{2.1$'$}$$

where $(x, j)_+ = (x, y)_+ = \sup\{j(x) \colon j \in J(x)\}$, $J(x) = \{j,\ j \in E^*,\ \|j\|^2 = \|x\| = j(x)\}$ (see [1]).

Then, we have

$$\|x_n - x_m\|^2 = (x_n - x_m, x_n - x_m)_+ = (x_n - fx_n + fx_n - fx_m + fx_m - x_m, x_n - x_m)_+$$
$$\leq \|x_n - x_m\| (\|x_n - fx_n\| + \|x_m - fx_m\|) + (fx_n - fx_m, x_n - x_m)_+;$$

if $\varepsilon > 0$, a $\nu \in N$ exists for which $n, m \geq \nu$ implies

$$\|x_n - x_m\|^2 \leq \varepsilon + g(\|x_n - x_m\|)\, \|x_n - x_m\|$$

since, $\alpha(\{x_n\}) = \alpha(\{x_n\}_{n \geq \nu})$, we obtain

$$\alpha^2(\{x_n\}) \leq g(\alpha(\{x_n\}))\, \alpha(\{x_n\}) + \varepsilon \quad \text{for all } \varepsilon > 0$$

and so

$$\alpha^2(\{x_n\}) \leq g(\alpha(\{x_n\}))\, \alpha(\{x_n\});$$

this fact implies $\alpha(\{x_n\}) = 0$ and the thesis is true like in (a_1).

(a_3) We suppose that there exists g, g: $\mathbb{R}^+ \to \mathbb{R}$, continuous with $g(r) < r$ if $r > 0$, for which (2.1) (or (2.1)$'$) is true.

Then, putting $v_{nm} = \|x_n - x_m\|$ and $v_r = \sup_{n,m \geq r} v_{n,m}$, we have like in ($a_2$):

$$v_{nm}^2 \leq \varepsilon + g(v_{nm}) v_{nm} \qquad \text{for all } n, m \geq \nu,\ \nu = \nu(\varepsilon).$$

Since there exists a sequence of v_{nm}'s which converges to v_r, $r \geq \nu(\varepsilon)$, we obtain

$$v_r^2 \leq \varepsilon + g(v_r) v_r \qquad \text{for all } r \geq \nu(\varepsilon)$$

On the other hand, $v_{r+1} \leqslant v_r$, $r \in N$, and so there is $v \geqslant 0$ for which $\lim_r v_r = v$. By using continuity of g, we have

$$v^2 \leqslant g(v)v + \varepsilon \qquad \text{for each } \varepsilon > 0;$$

then as in (a_2), we have $v = 0$. This fact implies that (i) and (ii) are true.

Now, we show that there exist f's which are nonexpansive and satisfy conditions like in (a_2) or (a_3), with an example.

We suppose $E = l_1$, $X = B(0, 1)$. We consider a function h, $h: X \to X$, defined by $h: (x_1, x_2, \ldots, x_n, \ldots) \to (-x_1, -x_2, \ldots, -x_n, \ldots)$; we have

$$\|hx\| = \|x\| \qquad \text{for each } x \in X \tag{2.2}$$

$$\|hx - hy\| = \|x - y\| \qquad \text{for each } x, y \in X \tag{2.3}$$

$$(hx - hy, x - y)_+ \leqslant 0 \qquad \text{for each } x, y \in X \tag{2.4}$$

(for a definition of $(.\,,\,.)_+$ and its calculation in l_1, see [1]).

Then, let k, $k: X \to X$, be the following function

$$k: (x_1, x_2, \ldots, x_n, \ldots) \to \tfrac{1}{2}((1 - \|x\|), x_{p(1)}, x_{p(2)}, \ldots, x_{p(n)}, \ldots)$$

where p is an arbitrary bijection from N onto N.

We have

$$\|kx\| \leqslant \tfrac{1}{2} \qquad \text{for each } x \in X \tag{2.5}$$

$$\|kx - ky\| \leqslant \|x - y\| \qquad \text{for each } x, y \in X \tag{2.6}$$

now, we define f, $f: X \to X$, by

$$fx = (1/2)(hx + kx) \qquad \text{for all } x \in X;$$

by (2.2) and (2.5) it follows $fx \in X$, for each $x \in X$; by (2.3) and (2.6) it follows that f is nonexpansive in X; moreover, if $x = 0$ and $y = (1, 0, \ldots, 0, \ldots)$, we have $\|f0 - fy\| = \|0 - y\|$; furthermore, (2.4) and (2.6) imply that, for any $j \in J(x - y)$, for each $x, y \in X$

$$(fx - fy, j)_+ = (fx - fy, x - y)_+ \leqslant (1/2)\|x - y\|^2;$$

then, we have to consider $g(r) = (1/2)r$ in (a_2) and (a_3).

SECTION 3

Now, we consider conditions on X, E which guarantee the weak convergence of the Mann–Ishikawa sequence to a fixed point of f.

More precisely, we suppose that E satisfies the so-called Opial's condition, i.e. (see [5]) for all x^0

$$\liminf_n \|x_n - x^0\| < \liminf_n \|x_n - y\| \qquad \text{for each } y \neq x^0$$

for each sequence $\{x_n\}$ which converges weakly to x^0. If X is convex and weakly compact, we shall prove that (i) and (ii) are true.

By weak compactness of X, there exists $\{x_{k(n)}\}$ which converges weakly to a $y \in X$. With standard proof we show that $y = fy$. We suppose that $\{x_n\}$ doesn't converge weakly to y; then, there are $\{x_{h(n)}\}$ and $z \neq y$ such that $x_{h(n)} \xrightarrow{w} z$; then, $z = fz$. By (1.2) there are $d(y)$, $d(z) \geqslant 0$ for which

$$d(y) = \lim_n \|x_n - y\|, \qquad d(z) = \lim_n \|x_n - z\|.$$

If $d(y) \leqslant d(z)$, i.e.

$$\lim_n \|x_n - y\| \leqslant \lim_n \|x_n - z\| \tag{3.1}$$

we have an absurdum, since by (3.1) it follows

$$\lim_n \|x_{h(n)} - y\| \leqslant \lim_n \|x_{h(n)} - z\|$$

which isn't true; in a similar way, we show that cannot be $d(y) \geqslant d(z)$.

Then, $\{x_n\}$ have to converge weakly to y.

SECTION 4

With similar proof like in [6] we can prove the two following theorems which extend some results due to Petryshyn & Williamson to the case of Mann–Ishikawa sequences for nonexpansive mappings (we observe that in [6] quasi-nonexpansive functions are considered).

THEOREM 4.1. Let f, X, E and $\{x_n\}$ be as in section 1. We suppose that $\mathscr{F}(f) \neq \varnothing$. Then, $\{x_n\}$ converges strongly to a fixed point of f if and only if

$$\lim_n d(x_n, \mathscr{F}(f)) = 0.$$

THEOREM 4.2. Let f, X, E and $\{x_n\}$ be as in section 1. Then, $\{x_n\}$ converges strongly to a fixed point of f if and only if there is a compact subset K of X for which

$$\lim_n d(x_n, K) = 0.$$

We observe that it is possible to show a result similar to Theorem 1.2 of [6].

Now, we consider nonexpansive f's satisfying a condition like

"given $\varepsilon > 0$ and $y \in X$ there exists $\delta > 0$ such that for each $x \in X$ for which $\varepsilon \leqslant \|x - y\| \leqslant \varepsilon + \delta$ we have $\|fx - fy\| \leqslant \varepsilon - \delta$".

If we suppose $\mathscr{F}(f) \neq \varnothing$ for such a f we can prove that the Mann–Ishikawa sequence converges strongly to the (unique) fixed point of f. Indeed, if $0 < \varepsilon = \lim_n \|x_n - z\|$, $z = fz$, there is $\delta > 0$ for which $\varepsilon \leqslant \|x_n - z\| < \varepsilon + \delta$, for sufficiently large n, using (1.2). Then, we have $\|fx_n - z\| \leqslant \varepsilon - \delta$. Since (1.1) is true, there exists $\bar{n} \in N$ sufficiently large such that

$$\|x_{\bar{n}} - fx_{\bar{n}}\| + \|fx_{\bar{n}} - z\| < \varepsilon;$$

by (1.2) it follows that

$$\varepsilon = \lim_n \|x_n - z\| \leqslant \|x_{\bar{n}} - z\| \leqslant \|x_{\bar{n}} - fx_{\bar{n}}\| + \|fx_{\bar{n}} - z\| < \varepsilon$$

which isn't true. Then, $\varepsilon = 0$ and our thesis is proved.

A condition like the above one is satisfied if f is a G-contraction (see [3], p. 47).

SECTION 5

In the last section, we consider the following set

$$M = \{f; f\colon X \to X, f \text{ nonexpansive}\},$$

endowed with the metric

$$d(f, g) = \sup\{\|fx - gx\| : x \in X\};$$

it is known that (M, d) is a complete metric space.

It is known that there exist nonexpansive functions which are fixed point free; for these mappings the Mann–Ishikawa iterative process doesn't converge strongly. Nevertheless, we can prove the following result:

THEOREM 5.1. If there is $a \in \mathbb{R}$ for which

$$0 < a \leqslant t_n \qquad \text{for each } n \in N,$$

there exists a dense G_δ-subset $\bar{\bar{M}}$ of M such that, for each $g \in \bar{\bar{M}}$, (i) and (ii) hold.

Proof. First of all, we observe that, if $M^0 = \{f; f \in M, f$ is a contraction with constant of contractivity $k_f \in]0, 1[\}$, $\bar{M}^0 = M$ and (i), (ii) are true for any $f \in M^0$.

Now, we consider

$$M_1 = \bigcap_{n \in N} \bigcup_{f \in M^0} B(f, \vee(f, 1/n)),$$

where $B(h, r) = \{g, g \in M, d(h, g) < r\}$ and $\vee(f, 1/n)$ is a real number such that $b \vee (f, 1/n)/(1 - H_f) \leqslant 1/n$, where $H_f = 1 - a + ak_f < 1$.

If $f \in M_1$, for each $n \in N$, there exists $f_n \in M^0$ such that $d(f, f_n) \leqslant \vee(f_n, 1/n)$. If we put, for each $h \in M_1$,

$$x^h_{m+1} = (1 - t_m)x^h_m + t_m h x^h_m \qquad \text{for all } m \in N$$

we have, if $x^g_1 = x^{f_m}_1 = x_1 \in X$,

$$\|x^g_{m+1} - x^{f_m}_{m+1}\| \leqslant b \vee (f_n, 1/n) \sum_{i=0}^{m-1} H^i_{f_n} \qquad \text{for all } m \in N; \tag{5.1}$$

we observe that (5.1) can be showed easily by induction on m.

By (5.1) it follows that

$$\limsup_m \|x^g_{m+1} - x^{f_n}_{m+1}\| \leqslant 1/n.$$

Now, we observe that Vidossich (see [7]) has proved that there is a dense G_δ-subset M_2 of M such that $M_2 \supseteq M^0$ and

(b_1) each $g \in M_2$ has a unique fixed point x^g.

(b_2) the function $f \to x^f$, $x^f = fx^f$, is continuous from M_2 into X.

Then, we put $\bar{\bar{M}} = M_1 \cap M_2$; obviously, $\bar{\bar{M}}$ is a nonempty dense G_δ-subset of M ($\bar{\bar{M}} \supseteq M^0$) and (i) is true for any $g \in \bar{\bar{M}}$. We have only to show (ii). for this purpose, let $g \in \bar{\bar{M}}$; there exists a sequence $\{f_n\} \subseteq M^0$ for which (5.1) is true.

Then, we have

$$\|x_m^g - x^g\| \leqslant \|x_m^g - x_n^{f_m}\| + \|x_n^{f_m} - x^{f_n}\| + \|x^{f_n} - x^g\| \qquad \text{for all } m, n \in N.$$

Given $\varepsilon > 0$, there exists $\mu \in N$ for which $\|x^{f_n} - x^g\| \leqslant \varepsilon$, for each $n \geqslant \mu$; then, if $\bar{n} \geqslant \mu$, $\bar{n} \geqslant 1/\varepsilon$, one has

$$\limsup_m \|x_{m+1}^g - x^g\| \leqslant \limsup_m \|x_{m+1}^g - x^{f_{\bar{n}}}\| + \limsup_m \|x_{m+1}^{f_{\bar{n}}} - x^{f_{\bar{n}}}\| + \limsup_m \|x^{f_{\bar{n}}} - x^g\| \leqslant 2\varepsilon$$

for each $\varepsilon > 0$.

This fact implies that

$$\lim_m \|x_{m+1}^g - x^g\| = 0.$$

Then, the proof is complete.

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NOTES

We suppose that (2.1) is true for a g such that

(i) $g(r) < r$ if $r > 0$;
(ii) there exists

$$\lim_{r \to \bar{r}^+} g(r) \leqslant g(\bar{r}), \text{ for } \bar{r} \in \mathbb{R}^+.$$

Then, using a proof as in "D. W. Boyd & J. S. W. Wong—On nonlinear contractions—*Proc. Am. Math. Soc.* **20**, 458–464 (1969)", we can show that $\{x_n\}$ converges strongly to the unique fixed point of f.

We observe that in (a_2) and (a_3) we can use $(.,.)_-$ (see [1]) instead of $(.,.)_+$.

Another result about weak convergence can be obtained if E is strictly convex, X is weakly compact and convex and f satisfies the following conditions

(j) f is demiclosed, i.e. $y_n - fy_n \xrightarrow{s} \theta$, $y_{k(n)} \xrightarrow{w} y$ imply $y = fy$ (then $\mathcal{F}(f) \neq \emptyset$).
(jj) there is an increasing function φ, $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies $\varphi(0) = 0$, $\lim_{r \to +\infty} \varphi(r) = +\infty$ and such that

$$((i - f)(x) - (i - f)(y), x - y)_+ \geqslant [\varphi(\|x\|) - \varphi(\|y\|)][\|x\| - \|y\|]$$

where i denotes the identity mapping on E.

In this case, we have that $\mathcal{F}(f)$ is a singleton; and so, by demiclosedness, $x_n \xrightarrow{w} x$, $\{x\} = \mathcal{F}(f)$.

We observe that a function f satisfying (jj) is called φ-accretive (see "H. Brezis & M. Sibony—Methodes d'approximation et d'iteration pour les operateurs monotones—*Arch. Rat. Mech. An.* **28**, 59–82 (1967/68)").

Now, we suppose that E satisfies the following assumption

$$y_n \xrightarrow{w} y, \|y_n\| \to \|y\| \qquad \text{imply } y_n \xrightarrow{s} y$$

(see "K. Fan, I. Glicksberg—Some geometric properties of the spheres in a normed linear space—*Duke Math. J.* **25**, 553–568 (1958)").

Since $x_n - fx_n \xrightarrow{s} \theta$, we have $\|x_n\| \to \|x\|$, $\{x\} = \mathcal{F}(f)$ (see Lemme 2.1 by Brezis & Sibony, *op. cit.*). Moreover, as above, $x_n \xrightarrow{w} x$; so, $x_n \xrightarrow{s} x$.

In this way, we extend a result by Gwinner ("J. Gwinner—On the convergence of some iteration processes in uniformly convex Banach spaces—*Proc. Am. Math. Soc.* **71**, 29–35 (1978)").