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FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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1. INTRODUCTION

THE STUDY in fixed point theory has generally developed in three main directions: generalization of conditions which ensure existence, and, if possible, uniqueness, of fixed points; investigation of the character of the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$, where $T: X \to X$, X a complete metric space, is the map under consideration; study of the topoligical properties of the set of fixed points, whenever T has more than one fixed point. This note treats only some aspects of the first and second question, along a line followed by many other authors. We mention, in particular, De Blasi [3], Kannan [4], Opial [7], Reich [8–10].

More precisely we consider maps $T: X \to X$, which satisfy conditions of the type

$$d(Tx, Ty) \leqslant \varphi(d(x, y)) + \psi(d(Tx, x)) + \chi(d(Ty, y)) \quad \text{for each } x, y \in X,$$

and for these mappings we prove, under suitable hypotheses, existence and uniqueness of fixed points.

The paper consists of four sections. In Section 1 we prove a fundamental lemma. In Section 2 we use this in order to establish the main fixed point theorems. In Section 3 we obtain some well-known results as corollaries of our theorems. Further results are presented in Section 4.

1. Through all the paper, X denotes a complete metric space and T, $T: X \to X$, an asymptotically regular mapping (see [2]); i.e., a function satisfying $\lim_n d(T^n x, T^{n+1} x) = 0$ for each $x \in X$. Furthermore, we suppose that there exist three functions φ, ψ, χ, from $[0, +\infty[$ into $[0, +\infty[$, which satisfy the assumptions:

(I_1) $\varphi(r) < r$ if $r > 0$,
(I_2) there exists $\lim_{r \to \bar{r}+} \varphi(r) \leqslant \varphi(\bar{r})$ for each $\bar{r} \in [0, +\infty[$
(I_3) $\psi(0) = \chi(0) = 0$.

Moreover, we suppose that T, φ, ψ, χ satisfy the inequality

(I_4) $d(Tx, Ty) \leqslant \varphi(d(x, y)) + \psi(d(Tx, x)) + \chi(d(Ty, y))$ for each $x, y \in X$.

LEMMA. Under the above assumptions on X and T and if, in addition, ψ and χ are continuous at $r = 0$, then, for each $x \in X$, there exists $z \in X$ such that $\{T^n x\}_{n=0}^{\infty}$ converges to z.

Proof. Suppose that there exists $x \in X$ such that the sequence of iterates is not a Cauchy sequence. Then, following [9] there exist $\varepsilon > 0$, $\{m(j)\}_{j=0}^{\infty}$, $\{n(j)\}_{j=0}^{\infty}$ which satisfy the conditions

$$m(j) > n(j) \quad \text{for each } j \in N \tag{1.1}$$

$$\lim_j n(j) = +\infty \tag{1.2}$$

$$d(T^{m(j)}x, T^{n(j)}x) \geqslant \varepsilon \tag{1.3}$$

$$d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon. \tag{1.4}$$

Then, we have

$$\varepsilon \leqslant d(T^{m(j)}x, T^{n(j)}x) \leqslant d(T^{m(j)}x, T^{m(j)-1}x) + d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon + d(T^{m(j)}x, T^{m(j)-1}x)$$

which implies

$$\lim_j d(T^{m(j)}x, T^{n(j)}x) = \varepsilon. \tag{1.5}$$

On the other hand

$$d(T^{m(j)}x, T^{n(j)}x) \leqslant d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) + d(T^{m(j)+1}x, T^{n(j)+1}x) \leqslant d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) + \varphi(d(T^{m(j)}x, T^{n(j)}x)) + \psi(d(T^{m(j)+1}x, T^{m(j)}x)) + \chi(d(T^{n(j)+1}x, T^{n(j)}x))$$

that is

$$d(T^{m(j)}x, T^{n(j)}x) - \varphi(d(T^{m(j)}x, T^{n(j)}x)) \leqslant d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) + \psi(d(T^{m(j)+1}x, T^{m(j)}x)) + \chi(d(T^{n(j)+1}x, T^{n(j)}x))$$

and, letting $j \to +\infty$

$$\varepsilon - \lim_j \varphi(d(T^{m(j)}x, T^{n(j)}x)) \leqslant 0.$$

Hence, by (1.3) and (1.5) it follows that $\varepsilon = 0$, a contradiction. Since X is a complete metric space, the proof is complete.

2. In this section we shall prove two fixed point theorems: the first for noncontinuous, the second for continuous mappings.

THEOREM 1. Let X, T, φ, ψ, χ be as in the Lemma. Furthermore, we suppose that $\chi(r) < r$ if $r > 0$. Then T has a unique fixed point.

Proof. Uniqueness is obvious by virtue of hypotheses (I_1), (I_3) and (I_4). So let us show existence. From the Lemma there is a $z \in X$ such that $T^n x \to z$, as $n \to +\infty$, for each $x \in X$. Since

$$d(z, Tz) \leqslant d(z, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tz) \leqslant d(z, T^n x) + d(T^n x, T^{n+1}x) + \varphi(d(T^n x, z)) + \chi(d(Tz, z)) + \psi(d(T^n x, T^{n+1}x))$$

we have

$$d(z, Tx) - \chi(d(z, Tz)) \leqslant 2d(z, T^n x) + d(T^n x, T^{n+1}x) + \psi(d(T^n x, T^{n+1}x))$$

and, letting $n \to +\infty$, we have

$$0 \leqslant d(z, Tz) - \chi(d(z, Tz)) \leqslant 0;$$

thus, $z = Tz$ follows.

Remark 1. Obviously, the role of the functions ψ and χ can be reversed.

For continuous functions, we can prove the following result.

THEOREM 2. Let $X, T, \varphi, \psi, \chi$ be as in the Lemma. If, in addition, T is continuous, then it has a unique fixed point.

Proof. Uniqueness follows as in Theorem 1. Let $z \in X$ be such that $T^n x \to z$, as $n \to +\infty$. From continuity of T we obtain

$$\lim_n T^{n+1}x = Tz$$

and, since T is asymptotically regular, we have $z = Tz$.

Remark 2. If X is a closed, non-void, subset of a Banach space B, Theorem 2 is still true under the assumption: "T is a strongly–weakly continuous mapping from X into X".

Remark 3. From the Lemma, Theorem 1 and Theorem 2, there follows that the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T, for each $x \in X$.

Remark 4. In [3] the author puts the following question: "Let T be an asymptotically regular, continuous mapping and let S be a non-void, weakly closed subset of a Hilbert space X. If $T, T: S \to S$, satisfies the condition

$$\|Tx - Ty\| \leqslant p\|x - y\| + q(\|Tx - x\| + \|Ty - y\|), \quad p^2 + q^2 = 1, \quad p, q \neq 0,$$

for each $x, y \in S$, does the sequence $\{T^n x\}_{n=0}^{\infty}$ converge to the unique fixed point of T, if it exists?"

By using Theorem 2, with $\varphi(r) = pr$, $\psi(r) = \chi(r) = qr$ for each $r \in [0, +\infty[$, we answer, positively, to this question. Furthermore, if we use Theorem 1, we can dispense with the continuity of T.

3. In this section we obtain some well-known results, as corollaries of our previous theorems.

COROLLARY 1 [6]. Let T be a function from X into X. We suppose that there exists a map f, f: $[0, +\infty[\to [0, +\infty[$, continuous from the right for each $r \in [0, +\infty[$, such that

$$d(Tx, Ty) \leqslant f(d(x, y)), \quad \text{for each } x, y \in X.$$

If $f(r) < r$, for $r > 0$, then the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

Proof. We use Theorem 2. In this theorem let $\varphi(r) = f(r), \psi(r) = \chi(r) = 0$ be, for each $r \in [0, +\infty[$. Moreover, if $v \in N$ exists such that $T^{v+1}x = T^v x$, for each $x \in X$, T is a asymptotically regular mapping. On the contrary, one has, for each $x \in X$ such that this v doesn't exist,

$$d(T^{n+1}x, T^n x) \leqslant f(d(T^n x, T^{n-1}x)) < d(T^n x, T^{n-1}x) \quad \text{for each } n \in N.$$

Put $d = \lim_n d(T^{n+1}x, T^n x)$, we have $d \leqslant f(d) \leqslant d$ and so $d = 0$.

COROLLARY 2 [1]. Let X, T be as in Corollary 1. Suppose that f, from $[0, +\infty[$ into $[0, +\infty[$ is a non-decreasing function, continuous from the right such that

$$d(Tx, Ty) \leqslant f(d(x, y)), \quad \text{for each } x, y \in X.$$

If $f(r) < r$ $(r > 0)$ and if X is bounded, then T has a unique fixed point.

CORALLARY 3 [3]. Let T be an asymptotically regular and continuous function, such that T maps S into S, with S a non-void weakly closed subset of a Hilbert space H. Also, we suppose that T satisfies

$$\|Tx - Ty\| \leqslant \|Tx - x\| + \|Ty - y\|, \quad \text{for each } x, y \in X.$$

Then, for each $x \in S$, the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

Proof. In Theorem 2, we take $\varphi(r) = 0$, $\psi(r) = \chi(r) = r$ for each $r \geqslant 0$.

COROLLARY 4 [3]. Let T be, $T: S \to S$, S a non-void weakly closed subset of a Hilbert space H, an asymptotically regular and continuous mapping, such that

$$\|Tx - Ty\| \leqslant p\|x - y\| + q(\|Tx - x\| + \|Ty - y\|), \quad \text{for each } x, y \in X,$$

with $p^2 + q^2 < 1$. Then, the thesis of Corollary 3 is true.

Proof. In Theorem 1, we put $\varphi(r) = pr, \psi(r) = \chi(r) = qr$, for each $r \geqslant 0$. We observe that, by using Theorem 1, we can dispense with the continuity of T.

COROLLARY 5 [8]. Let (X, d) be a complete metric space and let a, b, c be nonnegative numbers, with $a + b + c < 1$. Furthermore, suppose that $T: X \to X$ satisfies

$$d(Tx, Ty) \leqslant ad(x, y) + bd(Tx, x) + cd(Ty, y), \quad \text{for each } x, y \in X.$$

Then, T has a unique fixed point.

Proof. We can take, in Theorem 1, $\varphi(r) = ar$, $\psi(r) = br$, $\chi(r) = cr$, for each $r \in [0, +\infty[$. Since, for each $n \in N$, we have

$$d(T^{n+1}x, T^n x) \leqslant ad(T^n x, T^{n-1}x) + bd(T^{n+1}x, T^n x) + cd(T^n x, T^{n-1}x), \quad x \in X,$$

if follows that

$$d(T^{n+1}x, T^n x) \leqslant \left(\frac{a + c}{1 - b}\right)^n d(Tx, x), \quad x \in X.$$

This fact implies that T is asymptotically regular, being $a + b + c < 1$.

Remark 5. Under the assumptions of Corollary 5, we have, by virtue of Remark 3, that the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

COROLLARY 6 [5]. Let T, $T: X \to X$, be a mapping satisfying the condition

$$d(Tx, Ty) \leqslant (x, y) - \Delta(d(x, y)), \quad \text{for each } x, y \in X$$

where $\Delta, \Delta: [0, +\infty[\to [0, +\infty[$, is a continuous function such that $\Delta(r) > 0$, if $r > 0$. Then, for each $x \in X$, the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

Proof. Asymptotical regularity follows as in Corollary 1. By putting, in Theorem 1, $\phi(r) = r - \Delta(r)$, $\psi(t) = \chi(r) = 0$ for each $r \in [0, +\infty[$ one obtains the thesis. We observe that we can dispense with the continuity of Δ. In fact it suffices that Δ is a right continuous or a monotonically nondecreasing function.

4. In this section, we consider mappings T, $T: X \to X$, satisfying all assumptions of *n*. 1, with the exception of (I_4), which is replaced with

(I_4') $d(Tx, Ty) \leqslant \varphi(d(x, y)) + p(d(x, y))\,\psi(d(Tx, x)) + q(d(x, y))\,\chi(d(Ty, y))$ for each $x, y \in X$.
Here $p, p: [0, +\infty[\to [0, +\infty[$, is a function such that

(I_5) $\lim_{s \to \bar{s}+} p(s) < +\infty$, *for* each $\bar{s} \in [0, +\infty[$
and $q, q: [0, +\infty[\to [0, +\infty[$, is a function such that

(I_6) $\lim_{s \to 0+} q(s) < 1$ and $\lim_{s \to \bar{s}+} q(s) < +\infty$, for each $\bar{s} \in [0, +\infty[$.

Before proving the announced fixed point theorems, we observe that in this case the Lemma is still true if we suppose that ψ, χ are continuous functions as $r \to 0+$. Then, with standard arguments, we prove

THEOREM 3. If $X, T, \varphi, \psi, \chi, p, q$ satisfy all previous assumptions and, in addition, $\chi(r) \leqslant r$ for each $r \in [0, +\infty[$, then T has a unique fixed point. Moreover, the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

This last Theorem 3 generalizes the following result of [10].
"Let (X, d) be a complete metric space. If T, $T: X \to X$, satisfies

$$d(Tx, Ty) \leqslant a(d(x, y))\,d(x, y) + (b(x, y))\,d(Tx, x) + c(d(x, y))\,d(Ty, y), \tag{4.1}$$

where $x, y \in X, x \neq y$, and a, b, c are monotonically decreasing functions from $[0, +\infty[$ into $[0, 1[$ such that $a(s) + b(s) + c(s) < 1$, the T has a unique fixed point".
To this end, we observe, first of all, that T is an asymptotically regular mapping (see [10]). Moreover, we can take

$$\varphi(r) = \begin{cases} 0 & r = 0 \\ a(r)r & r \neq 0 \end{cases}; \quad \psi(r) = \chi(r) = r \quad \text{for each } r \in [0, +\infty[; \quad p(s) = 1$$

for each $s \in [0, +\infty[$; since we may assume $c(s) < \frac{1}{2}$ (see [9])

$$q(s) = \begin{cases} \frac{1}{2} & s = 0 \\ c(s) & s \neq 0 \cdot \end{cases}$$

By using Theorem 3 we obtain that T has a unique fixed point in X and, furthermore, that the sequence of iterates coverges to this unique fixed point.

Finally, if T is a continuous mapping satisfying assumptions as in the Lemma we may prove a result as Theorem 2, without assumptions $\lim_{s \to 0+} p(s) < +\infty$ and $\lim_{s \to 0+} q(s) < 1$.

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