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Gelfand-Phillips Property in a Banach Space of Vector Valued Measures

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In this paper (S, Σ, μ) will always denote a finite measure space and X a Banach space. $K(\mu, X)$ will denote the Banach space of μ-continuous X-valued measures G defined on Σ having relatively compact range, endowed with the semivariation norm (see [4], p. 223).

Purpose of this note is to show that if X is a Banach space having weak* sequentially compact dual balls or a separably complemented Banach space, then $K(\mu, X)$ has the socalled Gelfand-Phillips *property* (or $K(\mu, X)$ is a Gelfand-Phillips *space*), i.e. any limited set in $K(\mu, X)$ is a relatively compact set; we recall that a (bounded) set A in a Banach space E is said to be limited if for any $(x_n^*) \subset E^*$, $x_n^* \xrightarrow{w^*} \vartheta$, we have $\lim_n \sup_{x \in A} |x_n^*(x)| = 0$.

This class of Banach spaces has been recently investigated by Bourgain and Diestel in [1].

In order to obtain the results above mentioned, we have to use some lemmas.

Lemma 1 ([1]). *A subset A of a* Banach *space E is limited iff $T(A)$ is relatively compact in c_0 for any operator (i.e. bounded and continuous mapping) $T: E \to c_0$.*

Other useful Lemmas are

Lemma 2. *Let E and F be two isometrically isomorphic* Banach *spaces. If F is a* Gelfand-Phillips *space, then E is.*

Proof. The proof is easily obtained by Lemma 1.

Lemma 3. *If E is a* Gelfand-Phillips *space, then any closed subspace of E is.*

Proof. The proof is easily obtained by Lemma 1.

Lemma 4 ([3]). *Let H be a sequentially compact and compact* Hausdorff *topological space. Then, the usual* Banach *space $C(H)$ of real valued continuous functions on H is a* Gelfand-Phillips *space.*

Our first result can be now showed.

Theorem 1. *Let X be a* Banach *space having weak* sequentially compact dual balls. Then, $K(\mu, X)$ is a* Gelfand-Phillips *space.*

Proof. At first, we show that $K(\mu, X)$ can be isometrically embedded in a suitable Banach space of continuous functions. To this aim, let B_1 be the unit ball of $L^\infty(S, \Sigma, \mu) = (L^1(S, \Sigma, \mu))^*$ and B_2 be the unit ball of X^*.

Obviously, B_1 and B_2 are compact, sequentially compact HAUSDORFF topological spaces when endowed with their ω^*-topologies.

Hence, $H = B_1 \times B_2$ with the product topology satisfies the assumption of Lemma 4. We shall show that $K(\mu, X)$ can be isometrically embedded in $C(H)$, which is a GELFAND-PHILLIPS space by Lemma 4; an appeal to Lemma 3 and Lemma 2 will conclude our proof.

In [4], p. 223, it is shown that $K(\mu, X)$ is isometrically isomorphic to the completion of the PETTIS function space $P_1(\mu, X)$; moreover, J. K. BROOKS and N. DINCULEANU (see [2], Theorem 1) showed that $P_1(\mu, X)$ can be isometrically embedded in $C(H)$; hence, we can assume that $K(\mu, X)$ is a closed subspace of $C(H)$. This is the desired embedding. The proof is complete.

The above result will be used in order to prove the second theorem of this note. In the next result we use the concept of separably complemented BANACH space, i.e. a BANACH space X such that any separably closed subspace Y of X is contained in a complemented separable closed subspace Z of X. We recall that weakly compactly generated BANACH spaces have this property; and so even any BANACH lattice with order continuous norm is separably complemented ([5]).

Theorem 2. *Let X be a separably complemented* BANACH *space. Then, $K(\mu, X)$ is a* GELFAND-PHILLIPS *space.*

Proof. Let A be a limited set in $K(\mu, X)$. If (G_n) is a sequence in A, we shall prove that (G_n) has a convergent subsequence. Any G_n has a relatively compact range, hence there exists a closed separable subspace Y of X such that $G_n(S') \in Y$ for any $S' \in \Sigma$ and any $n \in N$. The above definition says that there exists a complemented separable closed subspace Z of X containing Y. Let $\tilde{P}$ be the projection of X onto Z; it determines a projection P of $K(\mu, X)$ onto $K(\mu, Z)$ in an obvious manner.

Moreover, we observe that $K(\mu, Z)$ is a GELFAND-PHILLIPS space by Theorem 1 and $(G_n) \subset K(\mu, Z)$.

Now, if $\tilde{T}$ is an operator from $K(\mu, Z)$ into c_0, we can define an operator T from $K(\mu, X)$ into c_0 by $T(G) = \tilde{T}(PG)$.

We have $T(A) \supset \{\tilde{T}(PG_n) : n \in N\}$; Lemma 1 gives that $(T(A)$ and hence$)$ $\{\tilde{T}(PG_n) : n \in N\}$ is relatively compact in c_0, for any operator $\tilde{T} : K(\mu, Z) \to c_0$. Hence the set $\{PG_n : n \in N\}$ is relatively compact in $K(\mu, Z)$; since $PG_n = G_n$, $n \in N$, we conclude the proof.

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