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A Linear Hyperbolic System and an Optimal Control Problem

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Abstract. We prove a theorem of existence, uniqueness, and continuous dependence for a linear hyperbolic system with Darboux-type conditions under assumptions on the coefficients, which are in a sense the most general possible. Moreover, an application of this result to an optimal control problem is given.

Key Words. Linear hyperbolic systems, Darboux-type conditions, existence and uniqueness theorems, continuous dependence theorems, minimization problems, existence theorems.

1. Introduction

In the present paper, we consider the following linear hyperbolic system (*state equation*):

$$\text{(E)}\quad z_{xy}+A(x,y)z_x+B(x,y)z_y+C(x,y)z = F(x,y)U(x,y)+G(x,y), \qquad \text{a.e. in } \Delta,$$

where

$$\Delta =]0,a[\times]0,b[, \qquad a,b>0,$$
$$z(x,y)\in\mathbb{R}^n, \qquad A(x,y),\ B(x,y),\ C(x,y)\in\mathbb{R}^{n,n},$$
$$F(x,y)\in\mathbb{R}^{n,m}, \qquad G(x,y)\in\mathbb{R}^n,$$

and the control U belongs to a given set

$$\mathcal{U}\subseteq L^p(\Delta,\mathbb{R}^m), \qquad p\in]1,+\infty[.$$

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Let the trace of z on the two sides of Δ which contain the origin be the *initial state*, and let a given subset Y of a suitable functional space (of Sobolev type) be the *target* to be achieved on the two sides of Δ which do not contain the origin.

For a given set X of initial states, we denote by $(X \times \mathscr{U})_a^Y$ the set of *admissible pairs*, i.e., the elements of $X \times \mathscr{U}$ for which the *output* z is such that $(z(\cdot, b), z(a, \cdot)) \in Y$.

In this framework, some rather general optimal control problems have been studied using direct methods of the calculus of variations (see, for example, Refs. 1–2). In the above quoted works, the coefficients A, B, C are assumed to be continuous in $\bar{\Delta}$, and a representation formula for the solutions of (E) is largely used. Also, Suryanarayana (Ref. 3) studies similar optimization problems, using different techniques, under boundedness assumptions for A, B, C.

In a recent paper (Ref. 4), the results of Ref. 2 are extended to the case of nonnecessarily bounded A, B, C. This is established by giving an existence, uniqueness, and continuous-dependence (on the control and the initial state) theorem for the solutions of (E) and using no representation formulas. More precisely, the assumptions on A, B, C, considered in Ref. 4, are as follows.

(a) A is measurable in Δ and there exists $\alpha \in L^p(]0, b[)$ such that $|A(x, y)| \leq \alpha(y)$, a.e. in Δ.

(b) B is measurable in Δ and there exists $\beta \in L^p(]0, a[)$ such that $|B(x, y)| \leq \beta(x)$, a.e. in Δ.

(c) $C \in L^p(\Delta)$.

Following this line, in the present paper, we improve further the hypotheses on A, B, C. Indeed, our assumptions are in a certain sense (see Theorem 2.2) the most general possible.

In particular, in section 2, we make some needed comments on the functional space of the solutions of the *control process* (E). In section 3, we prove an existence, uniqueness, and continuous-dependence theorem for the solutions of a boundary-value problem for a linear hyperbolic operator connected with (E). In section 4, we establish an existence theorem for the minimum in $(X \times \mathscr{U})_a^Y$ of a real functional J which depends on the initial state and the control, both in an explicit way and through the response $z(\cdot\,; (\varphi, \psi), U)$ of (E) and the derivatives of such response.

2. Some Properties of the Functional Space $W_p^*(R, \mathbb{R}^n)$

Henceforth, we shall assume $p \in]1, +\infty[$. Let $\Omega \subseteq \mathbb{R}^2$ be an open set. We denote by $W_p^*(\Omega, \mathbb{R}^n)$, see Refs. 5–6, the Banach space (of Sobolev type) of functions $w:(x,y)\to w(x,y)$, from Ω to $\mathbb{R}^n$, which belong to $L^p(\Omega,\mathbb{R}^n)$, together with their weak derivatives w_x, w_y, w_{xy}, endowed with the norm

$$\|w\|_{W_p^*(\Omega,\mathbb{R}^n)}=(\|w\|^p_{L^p(\Omega,\mathbb{R}^n)}+\|w_x\|^p_{L^p(\Omega,\mathbb{R}^n)}+\|w_y\|^p_{L^p(\Omega,\mathbb{R}^n)}+\|w_{xy}\|^p_{L^p(\Omega,\mathbb{R}^n)})^{1/p}.$$

In the case where Ω is an open rectangle of $\mathbb{R}^2$,

$$\Omega=R=]x_0,x_1[\times]y_0,y_1[,$$

one has the following characterization of $W_p^*(R,\mathbb{R}^n)$; see Ref. 5.

Theorem 2.1. A function w belongs to $W_p^*(R,\mathbb{R}^n)$, if it is of the form

$$w(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}h(\xi,\eta)\,d\xi\,d\eta+\int_{x_0}^{x}h_1(\xi)\,d\xi+\int_{y_0}^{y}h_2(\eta)\,d\eta+\gamma,\qquad \forall(x,y)\in R, \tag{1}$$

with

$$h\in L^p(R,\mathbb{R}^n),\qquad h_1\in L^p(]x_0,x_1[,\mathbb{R}^n),$$
$$h_2\in L^p(]y_0,y_1[,\mathbb{R}^n),\qquad \gamma\in\mathbb{R}^n.$$

Moreover, let $\mathscr{S}_p(R,\mathbb{R}^n)$ denote the product Banach space

$$\mathscr{S}_p(R,\mathbb{R}^n)=L^p(R,\mathbb{R}^n)\times L^p(]x_0,x_1[,\mathbb{R}^n)\times L^p(]y_0,y_1[,\mathbb{R}^n)\times\mathbb{R}^n.$$

Then, the following fact also holds (see Ref. 5).

Proposition 2.1. The transformation which maps each $(h,h_1,h_2,\gamma)\in\mathscr{S}_p(R,\mathbb{R}^n)$ into the element $w\in W_p^*(R,\mathbb{R}^n)$, defined by (1), is an algebraic and topological isomorphism between $\mathscr{S}_p(R,\mathbb{R}^n)$ and $W_p^*(R,\mathbb{R}^n)$.

Using Theorem 2.1 and Proposition 2.1, the following properties of $W_p^*(R,\mathbb{R}^n)$ can be easily established.

Proposition 2.2. $W_p^*(R,\mathbb{R}^n)$ is embedded both algebraically and topologically in $C^0(\bar{R},\mathbb{R}^n)$. Furthermore, the embedding is compact.

Proposition 2.3. Let $w\in W_p^*(R,\mathbb{R}^n)$. Then, for each $x\in[x_0,x_1]$ [resp., $y\in[y_0,y_1]$], one has

$$w(x,\cdot)\in W^{1,p}(]y_0,y_1[,\mathbb{R}^n)\qquad[\text{resp., } w(\cdot,y)\in W^{1,p}(]x_0,x_1[,\mathbb{R}^n)].$$

Moreover, for each $x \in [x_0, x_1]$ [resp., $y \in [y_0, y_1]$], the linear mapping (trace mapping) $w \to w(x, \cdot)$ [resp., $w \to w(\cdot, y)$], from $W_p^*(R, \mathbb{R}^n)$ to $W^{1,p}(]y_0, y_1[, \mathbb{R}^n)$ [resp., $W^{1,p}(]x_0, x_1[, \mathbb{R}^n)$] is continuous, uniformly with respect to $x \in [x_0, x_1]$ [resp., $y \in [y_0, y_1]$].

Proposition 2.4. Let $w \in W_p^*(R, \mathbb{R}^n)$. Then,

$$w_x(\bar{\xi}, \cdot) \in W^{1,p}(]y_0, y_1[, \mathbb{R}^n) \qquad [\text{resp., } w_y(\cdot, \bar{\eta}) \in W^{1,p}(]x_0, x_1[, \mathbb{R}^n)],$$

for a.e. $\bar{\xi} \in]x_0, x_1[$ [resp., $\bar{\eta} \in]y_0, y_1[$]. Moreover, the real function

$$\xi \to \|w_x(\xi, \cdot)\|_{C^0([y_0,y_1],\mathbb{R}^n)}, \qquad \xi \text{ a.e. in }]x_0, x_1[,$$

$$[\text{resp., } \eta \to \|w_y(\cdot, \eta)\|_{C^0([x_0,x_1],\mathbb{R}^n)}, \qquad \eta \text{ a.e. in }]y_0, y_1[],$$

is an element $\varepsilon(w)$ [resp., $\omega(w)$] of $L^p(]x_0, x_1[)$ [resp., $L^p(]y_0, y_1[)$]. Also, the linear mapping $w \to \varepsilon(w)$ [resp., $w \to \omega(w)$], from $W_p^*(R, \mathbb{R}^n)$ into $L^p(]x_0, x_1[)$ [resp., $L^p(]y_0, y_1[)$], is continuous.

By Propositions 2.2 and 2.4, one is allowed to introduce in $W_p^*(R, \mathbb{R}^n)$ another norm $|\cdot|_{W_p^*(R,\mathbb{R}^n)}$, putting

$$|w|_{W_p^*(R,\mathbb{R}^n)} = |w|_{C^0(\bar{R},\mathbb{R}^n)} + \|\varepsilon(w)\|_{L^p(]x_0,x_1[)}$$
$$+ \|\omega(w)\|_{L^p(]y_0,y_1[)} + \|w_{xy}\|_{L^p(R,\mathbb{R}^n)}, \qquad \forall w \in W_p^*(R, \mathbb{R}^n).$$

The following fact is an immediate consequence of Propositions 2.2 and 2.4.

Proposition 2.5. The norms $\|\cdot\|_{W_p^*(R,\mathbb{R}^n)}$ and $|\cdot|_{W_p^*(R,\mathbb{R}^n)}$ are equivalent.

Next, let us consider the second-order linear hyperbolic differential operator P,

$$Pw = w_{xy} + A(x, y)w_x + B(x, y)w_y + C(x, y)w,$$

where A, B, C are functions from R into $\mathbb{R}^{n,n}$.

We have the following theorem.

Theorem 2.2. For $P(W_p^*(R, \mathbb{R}^n)) \subseteq L^p(R, \mathbb{R}^n)$, it is a necessary and sufficient condition that A, B, C are measurable in R and satisfy

$$\sup_{x \in]x_0,x_1[} \text{ess} \int_{y_0}^{y_1} |A(x, \eta)|^p \, d\eta < +\infty, \tag{2}$$

$$\sup_{y \in]y_0,y_1[} \text{ess} \int_{x_0}^{x_1} |B(\xi, y)|^p \, d\xi < +\infty, \tag{3}$$

$$C \in L^p(R, \mathbb{R}^{n,n}). \tag{4}$$

Also,

$$P(W_p^*(R, \mathbb{R}^n)) \subseteq L^p(R, \mathbb{R}^n)$$

implies that P is a linear and continuous operator from $W_p^*(R, \mathbb{R}^n)$ into[3] $L^p(R, \mathbb{R}^n)$.

Proof. Let

$$P(W_p^*(R, \mathbb{R}^n)) \subseteq L^p(R, \mathbb{R}^n).$$

If we put

$$w^{(1)}(x, y) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \qquad w^{(n)}(x, y) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \forall (x, y) \in R,$$

then

$$w^{(i)} \in W_p^*(R, \mathbb{R}^n), \qquad i = 1, \ldots, n.$$

Hence, it follows from the hypothesis that

$$Pw^{(i)} \in L^p(R, \mathbb{R}^n), \qquad i = 1, \ldots, n.$$

Since

$$C = (Pw^{(1)}, \ldots, Pw^{(n)}),$$

we obtain

$$C \in L^p(R, \mathbb{R}^{n,n}).$$

Now, let h be an arbitrary element of $L^p(]y_0, y_1[)$. Let

$$g(y) = \int_{y_0}^{y} h(t)\, dt, \qquad \forall y \in]y_0, y_1[,$$

and

$$v^{(i)}(x, y) = g(y) w^{(i)}(x, y), \qquad \forall (x, y) \in R, \qquad i = 1, \ldots, n.$$

Then, one has

$$v^{(i)} \in W_p^*(R, \mathbb{R}^n), \qquad i = 1, \ldots, n,$$

[3] It will be proved later (see Theorem 3.1) that, if

$$P(W_p^*(R, \mathbb{R}^n)) \subseteq L^p(R, \mathbb{R}^n),$$

then one actually has

$$P(W_p^*(R, \mathbb{R}^n)) = L^p(R, \mathbb{R}^n).$$

and hence

$$Pv^{(i)} \in L^p(R, \mathbb{R}^n), \qquad i = 1, \ldots, n.$$

Since

$$hB = (Pv^{(1)}, \ldots, Pv^{(n)}) - gC,$$

it follows that

$$hB \in L^p(R, \mathbb{R}^{n,n}).$$

Since $h \in L^p(]y_0, y_1[)$ is arbitrary, the function B is measurable. Furthermore,

$$h|B| \in L^p(R), \qquad \text{for each } h \in L^p(]y_0, y_1[).$$

Consequently, if we put

$$\beta(y) = \int_{x_0}^{x_1} |B(\xi, y)|^p \, d\xi, \qquad y \text{ a.e. in }]y_0, y_1[,$$

then, by choosing $h \equiv 1$, we obtain

$$\beta(y) < +\infty, \qquad y \text{ a.e. in }]y_0, y_1[.$$

Moreover,

$$\int_{y_0}^{y_1} |h(t)|^p \beta(t) \, dt < +\infty, \qquad \text{for each } h \in L^p(]y_0, y_1[),$$

and so (Ref. 7, Theorem 20.15, p. 348)

$$\sup_{y \in]y_0, y_1[} \text{ess} \, \beta(y) < +\infty,$$

i.e., we have (3). In a similar way, we can show that (2) also holds.

Conversely, let A, B, C be measurable functions from R to $\mathbb{R}^{n,n}$ which satisfy conditions (2), (3), (4). Then, for each $w \in W_p^*(R, \mathbb{R}^n)$, Aw_x, Bw_y, Cw are measurable in R and

$$\iint_R |A(\xi, \eta) w_x(\xi, \eta)|^p \, d\xi \, d\eta$$
$$\leq \left\{ \sup_{x \in]x_0, x_1[} \text{ess} \int_{y_0}^{y_1} |A(x, \eta)|^p \, d\eta \right\} \|\epsilon(w)\|^p_{L^p(]x_0, x_1[)}, \tag{5}$$

$$\iint_R |B(\xi, \eta) w_y(\xi, \eta)|^p \, d\xi \, d\eta$$
$$\leq \left\{ \sup_{y \in]y_0, y_1[} \text{ess} \int_{x_0}^{x_1} |B(\xi, y)|^p \, d\xi \right\} \|\omega(w)\|^p_{L^p(]y_0, y_1[)}, \tag{6}$$

$$\iint_R |C(\xi, \eta)w(\xi, \eta)|^p \, d\xi \, d\eta \leq \|C\|^p_{L^p(R,\mathbb{R}^{n,n})} \|w\|^p_{C^0(\bar{R},\mathbb{R}^n)}. \tag{7}$$

Hence,

$$P(W_p^*(R, \mathbb{R}^n)) \subseteq L^p(R, \mathbb{R}^n).$$

Moreover, (5), (6), (7) imply

$$\|Pw\|_{L^p(R,\mathbb{R}^n)} \leq \text{const}|w|_{W_p^*(R,\mathbb{R}^n)}, \qquad \forall w \in W_p^*(R, \mathbb{R}^n).$$

Consequently, the linear operator P, from $W_p^*(R, \mathbb{R}^n)$ into $L^p(R, \mathbb{R}^n)$, is continuous. □

3. Boundary-Value Problem for the Operator P

To simplify our presentation, we shall assume that

$$R = \Delta =]0, a[\times]0, b[.$$

Let $\Xi_p^{(n)}$ and $\sum_p^{(n)}$ be the Banach spaces defined as follows:

$$\Xi_p^{(n)} = \{(\varphi, \psi) \in W^{1,p}(]0, a[, \mathbb{R}^n) \times W^{1,p}(]0, b[, \mathbb{R}^n): \varphi(0) = \psi(0)\},$$

$$\textstyle\sum_p^{(n)} = \{(\chi, \rho) \in W^{1,p}(]0, a[, \mathbb{R}^n) \times W^{1,p}(]0, b[, \mathbb{R}^n): \chi(a) = \rho(b)\},$$

with the norm derived from the product Banach space

$$W^{1,p}(]0, a[, \mathbb{R}^n) \times W^{1,p}(]0, b[, \mathbb{R}^n).$$

The following is an immediate consequence of Proposition 2.3.

Proposition 3.1. Let $w \in W_p^*(\Delta, \mathbb{R}^n)$. Then,

$$(w(\cdot, 0), w(0, \cdot)) \in \Xi_p^{(n)} \quad \text{and} \quad (w(\cdot, b), w(a, \cdot)) \in \textstyle\sum_p^{(n)}.$$

Moreover, the linear mappings $w \to (w(\cdot, 0), w(0, \cdot))$, from $W_p^*(\Delta, \mathbb{R}^n)$ into $\Xi_p^{(n)}$, and $w \to (w(\cdot, b), w(a, \cdot))$, from $W_p^*(\Delta, \mathbb{R}^n)$ into $\sum_p^{(n)}$, are continuous.

Now, let

$$f \in L^p(\Delta, \mathbb{R}^n), \qquad (\varphi, \psi) \in \Xi_p^{(n)}$$

be given. Consider the following boundary-value problem (BVP):

$$\begin{aligned} & w \in W_p^*(\Delta, \mathbb{R}^n), \\ & Pw = f, \\ & (w(\cdot, 0), w(0, \cdot)) = (\varphi, \psi). \end{aligned} \tag{8}$$

The following theorem is the main result of the present section.

Theorem 3.1. Assume

$$P(W_p^*(\Delta, \mathbb{R}^n)) \subseteq L^p(\Delta, \mathbb{R}^n).$$

Then, for any $((\varphi, \psi), f) \in \Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^n)$, the BVP (8) has a unique solution

$$w = w((\varphi, \psi), f).$$

Furthermore, the solution map

$$((\varphi, \psi), f) \to w((\varphi, \psi), f)$$

is an algebraic and topological isomorphism between $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^n)$ and $W_p^*(\Delta, \mathbb{R}^n)$.

Proof. For any open rectangle

$$J \subset \mathbb{R}^2, \qquad J =]x_0, x_1[\times]y_0, y_1[,$$

we denote by $\mathscr{W}_p(J, \mathbb{R}^n)$ the Banach space of the functions w in $W^{1,p}(J, \mathbb{R}^n) \cap C^0(\bar{J}, \mathbb{R}^n)$, such that

$$\int_{x_0}^{x_1} \operatorname*{sup\,ess}_{y \in]y_0, y_1[} |w_x(\xi, y)|^p \, d\xi < +\infty,$$

$$\int_{y_0}^{y_1} \operatorname*{sup\,ess}_{x \in]x_0, x_1[} |w_y(x, \eta)|^p \, d\eta < +\infty,$$

equipped with the norm

$$\|w\|_{\mathscr{W}_p(J, \mathbb{R}^n)} = \|w\|_{C^0(\bar{J}, \mathbb{R}^n)} + \left[\int_{x_0}^{x_1} \operatorname*{sup\,ess}_{y \in]y_0, y_1[} |w_x(\xi, y)|^p \, d\xi\right]^{1/p} + \left[\int_{y_0}^{y_1} \operatorname*{sup\,ess}_{x \in]x_0, x_1[} |w_y(x, \eta)|^p \, d\eta\right]^{1/p}.$$

Moreover, we denote by L the following first-order linear differential operator:

$$Lw = A(x, y) w_x + B(x, y) w_y + C(x, y) w.$$

By virtue of Theorem 2.2, A, B, C are measurable in Δ and verify (2), (3), (4). It follows that

$$L(\mathscr{W}_p(J, \mathbb{R}^n)) \subseteq L^p(J, \mathbb{R}^n)$$

and

$$\|Lw\|_{L^p(J, \mathbb{R}^n)} \leq \mathscr{K} \|w\|_{\mathscr{W}_p(J, \mathbb{R}^n)}, \qquad \forall w \in \mathscr{W}_p(J, \mathbb{R}^n),$$

for each $J \subseteq \Delta$, where

$$\mathscr{K} = \left[\operatorname*{sup\,ess}_{x \in]0,a[} \int_0^b |A(x, \eta)|^p \, d\eta \right]^{1/p} + \left[\operatorname*{sup\,ess}_{x \in]0,b[} \int_0^a |B(\xi, y)|^p \, d\xi \right]^{1/p} + \|C\|_{L^p(\Delta, \mathbb{R}^{n,n})} \tag{9}$$

is a constant which does not depend upon J.

Now, we notice that the BVP (8) is equivalent to the integrodifferential equation

$$w(x, y) = \tilde{w}(x, y) - \int_0^x \int_0^y (Lw)(\xi, \eta) \, d\xi \, d\eta, \qquad \forall (x, y) \in \Delta, \tag{10}$$

in the unknown $w \in \mathscr{W}_p(\Delta, \mathbb{R}^n)$, where $\tilde{w} \in W_p^*(\Delta, \mathbb{R}^n)$, and hence $\tilde{w} \in \mathscr{W}_p(\Delta, \mathbb{R}^n)$, is given by

$$\tilde{w}(x, y) = \varphi(x) + \psi(y) - \varphi(0) + \int_0^x \int_0^y f(\xi, \eta) \, d\xi \, d\eta, \qquad \forall (x, y) \in \Delta.$$

Then, the first assertion of our theorem will be proved if we show that (10) has a unique solution. We proceed as follows.

Consider any sequence $\{w_k\} \subset \mathscr{W}_p(\Delta, \mathbb{R}^n)$ recursively defined as follows:

w_0 is any element of $\mathscr{W}_p(\Delta, \mathbb{R}^n)$,

$$w_k(x, y) = \tilde{w}(x, y) - \int_0^x \int_0^y (Lw_{k-1})(\xi, \eta) \, d\xi \, d\eta, \quad \forall (x, y) \in \Delta, \quad \forall k \in N. \tag{11}$$

We shall show that $\{w_k\}$ converges to a $w \in \mathscr{W}_p(\Delta, \mathbb{R}^n)$ in the norm topology of $\mathscr{W}_p(\Delta, \mathbb{R}^n)$. Such w will be a solution of (10), since convergence in $\mathscr{W}_p(\Delta, \mathbb{R}^n)$ implies uniform convergence and L is continuous. To this end, it is sufficient to show that

$$\textstyle\sum_{k=1}^{\infty} \|z_k\|_{\mathscr{W}_p(\Delta, \mathbb{R}^n)} < +\infty,$$

where

$$z_k = w_k - w_{k-1}, \qquad k \in N.$$

To do this, we divide Δ in smaller rectangles

$$\Delta_{ij} =]a_{i-1}, a_i[\times]b_{j-1}, b_j[, \qquad i = 1, \ldots, r, j = 1, \ldots, s,$$

with

$$0 = a_0 < a_1 < \cdots < a_r = a, \qquad 0 = b_0 < b_1 < \cdots < b_s = b,$$

and we put

$$|w|_{ij} = \begin{cases} 0, & \text{if} \quad ij = 0, \\ \|w\|_{\mathscr{W}_p(\Delta_{ij}, \mathbb{R}^n)}, & \text{if} \quad ij > 0, \end{cases}$$

for

$$i = 1, \ldots, r, \qquad j = 1, \ldots, s, \qquad w \in \mathscr{W}_p(\Delta, \mathbb{R}^n).$$

Since

$$z_{k+1}(x, y) = -\int_0^x \int_0^y (Lz_k)(\xi, \eta)\, d\xi\, d\eta, \qquad \forall (x, y) \in \Delta, \forall k \in N,$$

it follows that

$$\begin{aligned} &z_{k+1} = \tilde{z}_{ij,k} + \tau_{j,k} + \sigma_{i,k} + \omega_{ij,k}, \\ &i = 1, \ldots, r, \qquad j = 1, \ldots, s, \qquad k \in N, \end{aligned} \tag{12}$$

where $\tilde{z}_{ij,k}, \tau_{j,k}, \sigma_{i,k}, \omega_{ij,k} \in \mathscr{W}_p(\Delta, \mathbb{R}^n)$ are defined by

$$\tilde{z}_{ij,k}(x, y) = -\int_{a_{i-1}}^x \int_{b_{j-1}}^y (Lz_k)(\xi, \eta)\, d\xi\, d\eta,$$

$$\tau_{j,k}(x, y) = z_k(x, b_{j-1}), \qquad \sigma_{i,k}(x, y) = z_k(a_{i-1}, y),$$

$$\omega_{ij,k}(x, y) = -z_k(a_{i-1}, b_{j-1}),$$

for each

$$(x, y) \in \Delta, \qquad i = 1, \ldots, r, \qquad j = 1, \ldots, s, \qquad k \in N.$$

Moreover, it can be easily shown that

$$\begin{aligned} &|\tau_{j,k}|_{ij} \leq |z_k|_{ij-1}, \qquad |\sigma_{i,k}|_{ij} \leq |z_k|_{i-1j}, \\ &|\omega_{ij,k}|_{ij} \leq |z_k|_{i-1j-1}, \\ &i = 1, \ldots, r, \qquad j = 1, \ldots, s, \qquad k \in N. \end{aligned} \tag{13}$$

Furthermore, for

$$i = 1, \ldots, r, \qquad j = 1, \ldots, s, \qquad k \in N,$$

we have

$$|\tilde{z}_{ij,k}|_{ij} \leq \mu_p(\Delta_{ij}) \|Lz_k\|_{L^p(\Delta_{ij}, \mathbb{R}^n)} \leq \mu_p(\Delta_{ij}) \mathscr{K} |z_k|_{ij},$$

with

$$\mu_p(\Delta_{ij}) = [(a_i - a_{i-1})(b_j - b_{j-1})]^{1/p'} + (a_i - a_{i-1})^{1/p'} + (b_j - b_{j-1})^{1/p'},$$

$$1/p + 1/p' = 1,$$

and $\mathcal{K}$ given by (9). Obviously, the decomposition of Δ can be made in such a way that there exists $\lambda \in]0, 1[$ for which

$$|\tilde{z}_{ij,k}|_{ij} \leq \lambda |z_k|_{ij}, \qquad i=1,\ldots,r, j=1,\ldots,s, k\in N. \tag{14}$$

From (12), (13), (14), it follows that

$$|z_{k+1}|_{ij} \leq \lambda |z_k|_{ij} + |z_{k+1}|_{i-1j} + |z_{k+1}|_{ij-1} + |z_{k+1}|_{i-1j-1},$$

$$i=1,\ldots,r, \qquad j=1,\ldots,s, \qquad k\in N.$$

Hence, for

$$i=1,\ldots,r, \qquad j=1,\ldots,s,$$

we have

$$\sum_{k=1}^{\infty} |z_k|_{ij} < +\infty,$$

from which the result

$$\textstyle\sum_{k=1}^{\infty} \|z_k\|_{\mathcal{W}_p(\Delta,\mathbb{R}^n)} < +\infty$$

follows. This completes the proof of the existence of a solution of (10).

Next, we prove the uniqueness of such a solution as follows. We suppose that $w_1, w_2 \in \mathcal{W}_p(\Delta, \mathbb{R}^n)$ are solutions of (10). Let

$$z = w_1 - w_2.$$

By dividing Δ as above, we obtain

$$|z|_{ij} \leq \lambda |z|_{ij} + |z|_{i-1j} + |z|_{ij-1} + |z|_{i-1j-1},$$

$$i=1,\ldots,r, \qquad j=1,\ldots,s.$$

It follows that

$$|z|_{ij} = 0, \qquad i=1,\ldots,r, \qquad j=1,\ldots,s,$$

and so

$$w_1 = w_2.$$

Therefore, the solution of (10) is unique. We shall denote the solution by $w((\varphi,\psi),f)$.

Finally, we show that the linear mapping

$$((\varphi,\psi),f) \to w((\varphi,\psi),f)$$

is an algebraic and topological isomorphism between $\Xi^{(n)} \times L^p(\Delta,\mathbb{R}^n)$ and $W_p^*(\Delta,\mathbb{R}^n)$. Indeed, the assumption

$$P(W_p^*(\Delta,\mathbb{R}^n)) \subseteq L^p(\Delta,\mathbb{R}^n),$$

Proposition 2.3, the existence and uniqueness of $w((\varphi, \psi), f)$ imply that

$$((\varphi, \psi), f) \to w((\varphi, \psi), f)$$

is onto. Moreover, injectivity is trivial. So, we have only to prove the bicontinuity of our mapping. To this aim, it is sufficient to show that the inverse mapping, from $W_p^*(\Delta, \mathbb{R}^n)$ to $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^n)$, is continuous. But this is an immediate consequence of Proposition 2.3 and Theorem 2.2. This concludes the proof of our theorem. □

Remark 3.1. The argument used in the proof of Theorem 3.1 shows that the unique solution w of BVP (8) is the limit in the space $\mathcal{W}_p(\Delta, \mathbb{R}^n)$ of any sequence $\{w_k\}$ given by (11). We observe that

$$w_k \in W_p^*(\Delta, \mathbb{R}^n), \qquad k \in N.$$

Moreover,

$$\lim_k w_k = w$$

also in the space $W_p^*(\Delta, \mathbb{R}^n)$. Indeed, from

$$\lim_k w_k = w, \qquad \text{in } \mathcal{W}_p(\Delta, \mathbb{R}^n),$$

it follows that

$$\lim_k Lw_k = Lw, \qquad \text{in } L^p(\Delta, \mathbb{R}^n).$$

Hence, by Proposition 2.1,

$$\lim_k w_k = w, \qquad \text{in } W_p^*(\Delta, \mathbb{R}^n).$$

4. Existence Theorem for a Minimum Problem Related to the Control Process

Now, we return to the control process (E). In what follows, we assume the following.

(i) A, B, C are measurable functions from Δ to $\mathbb{R}^{n,n}$.

(ii) $\sup\text{ess}_{x\in]0,a[} \int_0^b |A(x, \eta)|^p \, d\eta < +\infty, \qquad \sup\text{ess}_{y\in]0,b[} \int_0^a |B(\xi, y)|^p \, d\xi < +\infty,$

$C \in L^p(\Delta, \mathbb{R}^{n,n})$.

(iii) $F \in L^\infty(\Delta, \mathbb{R}^{n,m}), \qquad G \in L^p(\Delta, \mathbb{R}^n)$.

Remark 4.1. Conditions (i), (ii) on A, B, C are strictly more general than (a), (b), (c), mentioned in Section 1 and considered in Ref. 4. Indeed, (a), (b), (c) imply (i), (ii). On the other hand, if

$$a=b=1, \qquad n=1,$$

$$A(x, y)=\begin{cases}(1/y)^{1/p}, & 0<x<1, x/2<y<x,\\ 0, & \text{elsewhere in }]0,1[\times]0,1[,\end{cases}$$

then A verifies

$$\sup_{x\in]0,1[} \operatorname{ess} \int_0^1 |A(x,\eta)|^p \, d\eta < +\infty,$$

whereas condition (a) does not hold.

Next, we make some preliminary observations. We notice that, from Theorem 3.1, the proposition below follows.

Proposition 4.1. For any $U \in L^p(\Delta, \mathbb{R}^m)$, $(\varphi, \psi) \in \Xi_p^{(n)}$, there exists in $W_p^*(\Delta, \mathbb{R}^n)$ a unique solution

$$z(x, y) = z(x, y; (\varphi, \psi), U)$$

of (E) satisfying the conditions

$$\text{(C)} \quad z(x, 0)=\varphi(x), \qquad x\in]0, a[, \qquad z(0, y)=\psi(y), \qquad y\in]0, b[.$$

Furthermore, the mapping

$$((\varphi, \psi), U) \rightarrow z(\cdot\,; (\varphi, \psi), U),$$

from $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^n)$ into $W_p^*(\Delta, \mathbb{R}^n)$, is affine and continuous.

Moreover, we observe that Propositions 2.3 and 3.1 imply the following proposition.

Proposition 4.2. The mapping

$$((\varphi, \psi), U) \rightarrow (z(\cdot\,, b; (\varphi, \psi), U), z(a, \cdot\, (\varphi, \psi), U))$$

is affine and continuous from $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^n)$ into $\sum_p^{(n)}$.

Now, let

$$X \subseteq \Xi_p^{(n)}, \qquad \mathcal{U} \subseteq L^p(\Delta, \mathbb{R}^m), \qquad Y \subseteq \textstyle\sum_p^{(n)},$$

and let $(X \times \mathcal{U})_a^Y$ denote the set of $((\varphi, \psi), U) \in X \times \mathcal{U}$, such that

$$(z(\cdot, b; (\varphi, \psi), U), z(a, \cdot; (\varphi, \psi), U)) \in Y.$$

Consider the following optimal control problem.

Problem (P). Let $(X \times \mathcal{U})_a^Y$ be nonempty. Minimize in $(X \times \mathcal{U})_a^Y$ the cost J defined in $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$ by

$$\begin{aligned} J((\varphi, \psi), U) = \iint_\Delta & H(\xi, \eta; z(\xi, \eta; (\varphi, \psi), U), z_x(\xi, \eta; (\varphi, \psi), U), \\ & z_y(\xi, \eta; (\varphi, \psi), U); U(\xi, \eta); \varphi(\xi), \varphi'(\xi), \psi(\eta), \psi'(\eta), \\ & z(\xi, b; (\varphi, \psi), U), z_x(\xi, b; (\varphi, \psi), U), z(a, \eta; (\varphi, \psi), U), \\ & z_y(a, \eta; (\varphi, \psi), U))\, d\xi\, d\eta, \\ & \forall((\varphi, \psi), U) \in \Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m), \end{aligned} \tag{15}$$

where

$$(x, y; z_1, z_2, z_3; u; v_1, \ldots, v_8) \to H(x, y; z_1, z_2, z_3; u, v_1, \ldots, v_8)$$

is a suitable function from $\Delta \times \mathbb{R}^{3n} \times \mathbb{R}^m \times \mathbb{R}^{8n}$ into $\mathbb{R}$.

More precisely, we shall assume the following hypotheses.

Assumption 4.1. H is measurable with respect to (x, y) for each $(z_1, z_2, z_3; u; v_1, \ldots, v_8)$ and is continuous with respect to $(z_1, z_2, z_3; u; v_1, \ldots, v_8)$ for a.e. (x, y).

Assumption 4.2. There exist

$$l \in L^1(\Delta), \qquad q \in C^0(\mathbb{R}^n \times \mathbb{R}^{4n}), \qquad b \in L^\infty(\Delta),$$

such that

$$\begin{aligned} & |H(x, y; z_1, z_2, z_3; u; v_1, \ldots, v_8)| \\ & \quad \le l(x, y) q(z_1; v_1, v_3, v_5, v_7) + b(x, y)\left[|z_2|^p + |z_3|^p + |u|^p + \sum_{i=1}^4 |v_{2i}|^p\right], \\ & \qquad \text{a.e. } (x, y) \in \Delta, \qquad \forall (z_1, z_2, z_3; u; v_1, \ldots, v_8) \in \mathbb{R}^{3n} \times \mathbb{R}^m \times \mathbb{R}^{8n}. \end{aligned}$$

Assumptions 4.1 and 4.2 imply the finiteness of the integral in (15) for each $((\varphi, \psi), U) \in \Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$.

Through the following, we shall consider also the following hypotheses.

Assumption 4.3. For a.e. $(x, y) \in \Delta$, the function

$$(z_1, z_2, z_3; u; v_1, \dots, v_8) \to H(x, y; z_1, z_2, z_3; u; v_1, \dots, v_8)$$

is convex.

Assumption 4.4. For a.e. $(x, y) \in \Delta$, for any $(z_1; v_1, v_3, v_5, v_7) \in \mathbb{R}^n \times \mathbb{R}^{4n}$, the function

$$(z_2, z_3; u; v_2, v_4, v_6, v_8) \to H(x, y; z_1, z_2, z_3; u; v_1, \dots, v_8)$$

is convex. Furthermore, there exist $\nu \in \mathbb{R}^+$, $\rho \in L^1(\Delta)$, such that

$$\begin{aligned}|H(x, y; z_1', z_2, z_3; u; v_1', v_2, v_3', v_4, v_5', v_6, v_7', v_8)\\ - H(x, y; z_1'', z_2, z_3; u; v_1'', v_2, v_3'', v_4, v_5'', v_6, v_7'', v_8)|\\ \leq \nu\Bigg[\rho(x, y) + |z_2|^p + |z_3|^p + |u|^p\\ + \sum_{i=1}^{4} |v_{2i}|^p)(|z_1' - z_1''| + \sum_{i=1}^{4} |v_{2i-1}' - v_{2i-1}''|^p\Bigg],\end{aligned}$$

for a.e. $(x, y) \in \Delta$, for each $(z_1', z_2, z_3; u; v_1', v_2, v_3', v_4, v_5', v_6, v_7', v_8)$, $(z_1'', z_2, z_3; u; v_1'', v_2, v_3'', v_4, v_5'', v_6, v_7'', v_8) \in \mathbb{R}^{3n} \times \mathbb{R}^m \times \mathbb{R}^{8n}$.

Assumption 4.5. There exist constants σ, σ_1, α, $\sigma > 0$, $\alpha \in [0, p]$, and a function $\sigma_2 \in L^1(\Delta)$, such that

$$H(x, y; z_1, z_2, z_3; u; v_1, \dots, v_8) \geq \sigma|u|^p + \sigma_1|u|^\alpha + \sigma_2(x, y),$$

for a.e. $(x, y) \in \Delta$, for any $(z_1, z_2, z_3; u; v_1, \dots, v_8) \in \mathbb{R}^{3n} \times \mathbb{R}^m \times \mathbb{R}^{8n}$.

Assumption 4.6. There exist constants ϑ, ϑ_1, β, $\vartheta > 0$, $\beta \in [0, p[$, and a function $\vartheta_2 \in L^1(\Delta)$, such that

$$H(x, y; z_1, z_2, z_3; u; v_1, \dots, v_8) \geq \vartheta \sum_{i=1}^{4} |v_i|^p + \vartheta_1 \sum_{i=1}^{4} |v_i|^\beta + \vartheta_2(x, y),$$

for a.e. $(x, y) \in \Delta$, for any $(z_1, z_2, z_3; u; v_1, \dots, v_8) \in \mathbb{R}^{3n} \times \mathbb{R}^m \times \mathbb{R}^{8n}$.

Now, we are ready to prove the following existence theorem for the Problem (P).

Theorem 4.1. Let X, $\mathcal{U}$, Y be weakly closed. Let H verify Assumptions 4.1, 4.2, and moreover either Assumption 4.3 or 4.4. Let X be bounded or otherwise let Assumption 4.6 hold. Let $\mathcal{U}$ be bounded or otherwise let Assumption 4.5 hold. Let $(X \times \mathcal{U})_a^Y \neq \varnothing$. Then, Problem (P) has a solution.

Proof. We start by showing that J is weakly sequentially lower semicontinuous in $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$. Indeed, if Assumption 4.3 is verified, then it can be shown that J is convex and continuous in $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$; see Ref. 4, Lemma 4.1. On the other hand, if Assumption 4.4 is verified, then the weak sequential lower semicontinuity of J is obtained as in Ref. 4 (Proposition 4.1) using a well-known theorem of Browder (Ref. 8, Theorem 2). Next, we consider a minimizing sequence

$$\{((\varphi_k, \psi_k), U_k)\} \subseteq X.$$

Any such a sequence must be bounded in any case (see Ref. 4, Proof of Theorem 4.1). Since $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$ is reflexive, then there exists a subsequence $\{((\varphi_{n_k}, \psi_{n_k}), U_{n_k})\}$, weakly convergent to $((\varphi, \psi), U)$ in $\Xi_p^{(n)} \times L^p(\Delta, \mathbb{R}^m)$. Since $(X \times \mathcal{U})_a^Y$ is weakly closed (Proposition 4.2), then

$$((\varphi, \psi), U) \in (X \times \mathcal{U})_a^Y.$$

From the weak sequential lower semicontinuity of J, it follows that $((\varphi, \psi), U)$ is a solution of Problem (P). □

Remark 4.2. The cost J, considered here, is more general than the one studied in Refs. 1–4, since H depends upon z_x, z_y, too. This fact allows us, for instance, to cover the following *minimal area problem*: Given

$$(\varphi, \psi) \in \Xi_p^{(1)}, \qquad \mathcal{U} \subseteq L^p(\Delta, \mathbb{R}^m), \qquad Y \subseteq \Xi_p^{(1)},$$

find, among the controls $U \in \mathcal{U}$ which steer (φ, ψ) into Y, a control U such that the corresponding surface

$$z = z(x, y; (\varphi, \psi), U), \qquad (x, y) \in \bar{\Delta},$$

has minimal area.

Remark 4.3. Finally, we notice that it is possible to use an intermediate hypothesis between Assumption 4.2 and Assumption 4.4, instead of Assumption 4.3 or Assumption 4.4, as in Ref. 4, Remark 4.2.

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