

SOME ISOMORPHIC PROPERTIES IN $K(X, Y)$ AND IN PROJECTIVE TENSOR PRODUCTS.

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ABSTRACT. We study the (DPrp) and the Gelfand Phillips property in the space of compact operators. Moreover we give some sufficient conditions in order that a projective tensor product of two Banach spaces is sequentially Right (SR) or it has the L-limited property. Moreover we study the Bourgain Diestel property (BD), the (RDP^*) property in the space $K_{w^*}(X, Y)$. We introduce the dual (SR^*) property and we give a characterization of it.

In this note we wish to collect some lifting results of certain isomorphic properties of Banach spaces to spaces of compact operators and to projective tensor products. In [15] the second author introduced the following

Definition 1. *A Banach space X has the (DPrp) if every Dunford Pettis subset of X is also relatively compact.*

Easily every Schur space has the (DPrp), since it is well known that Dunford Pettis sets are conditionally weakly compact (see, for instance, [15]). It is also known (see again [15]) that every dual Banach space has the (DPrp) if and only if it has the weak Radon Nikodym property that is if and only if its predual does not contain a copy of l_1 . In [15] and in [19] the authors studied the lifting of the (DPrp) from X^* , the dual space of a Banach space X , and from a Banach space Y to the space $K(X, Y)$ of all compact operators from X into Y . In this note we shall furnish a new condition in order that $K(X, Y)$ has the (DPrp). Such a result actually is strictly more general than the previous ones as shown with Example 4.

We shall also study the Gelfand Phillips property for closed subsets of the space $\mathcal{K}(X, Y)$ (we refer the reader to [32] for more about this property).

We shall also investigate another property, quite recently introduced in [31], i.e. the so called sequentially Right (in short (SR)) property. From the results in [25] it follows that the (SR) property is an intermediate property between two other very famous isomorphic properties, the Pelczynski property (V) ([30]; see below for the definition) and the Reciprocal Dunford Pettis (RDP) property ([18]; see below for the definition). In this note we give a sufficient condition in order that the projective tensor product of two Banach spaces can be (SR). Again, we shall relate the sequential right property with the L-limited property introduced in [29]. Moreover, following the way drawn by Pelczynski ([30]), we introduce the (SR^*) property that is a dual property with respect to (SR) property and we give a characterization of it.

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At the end, we shall study the Bourgain Diestel property (in short (BD)) and the (RDP^*) property (see Definition 29 below) in the space $K_{w^*}(X^*, Y)$ of all $w^* - w$ continuous compact operators from X^* into Y .

Our notations are standard. Throughout X, Y, E, F denote Banach spaces, X^* is the dual of X , and B_X stands for its closed unit ball. The closed unit ball B_{X^*} will always be endowed with the weak-star topology. By \mathbb{N} we represent the set of all natural numbers.

A bounded subset K in a Banach space X is limited (respectively Dunford Pettis) if

$$\limsup_n \sup_{x \in K} |x_n^*(x)| = 0$$

for every weak* null (respectively weakly null) sequence (x_n^*) in X^* .

Also if $A \subset X^*$ and every weak null sequence (x_n) in X converges uniformly on A , we say that A is an L-set. Easily every relatively compact subset of X is limited and clearly every limited set is a Dunford Pettis set and every Dunford Pettis subset of a dual Banach space is an L-set, but the converse of these assertions, in general, are false.

We use the symbol $\mathcal{L}(X, Y)$ for the space of all (linear bounded) operators from X into Y endowed with the operator norm, while $\mathcal{K}(X, Y), \mathcal{W}(X, Y), \mathcal{DP}(X, Y), \mathcal{Lcc}(X, Y)$ denote the subspaces of all compact, weakly compact, Dunford Pettis (or completely continuous), limited-completely continuous operators respectively, where an operator $T : X \rightarrow Y$ is said to be Dunford Pettis (or completely continuous) if it maps weakly Cauchy sequences into norm convergent sequences, whereas it is called limited-completely continuous if it sends limited weakly null sequences into norm null sequences.

A Banach space X has the Dunford-Pettis property (in short (DP) property) if, for any Banach space Y , every weakly compact operator $T : X \rightarrow Y$ is completely continuous. This is equivalent to saying that, for any weakly null sequences (x_n) and (x_n^*) in X and X^* respectively, $\lim_n |x_n^*(x_n)| = 0$ (see, e.g., [9][Theorem 1]).

We refer to [9] for more information on the Dunford-Pettis property.

In [30] Pelczynski introduced the property (V): a Banach space X has the property (V) if for every Banach space Y , every unconditionally converging operator $T : X \rightarrow Y$, i.e. an operator mapping weakly unconditionally converging series onto unconditionally converging ones, is weakly compact, whereas in [18] Grothendieck introduced the Reciprocal Dunford-Pettis, in symbols (RDP), property: a Banach space X has the (RDP) property if for every Banach space Y , every completely continuous operator $T : X \rightarrow Y$ is weakly compact.

Every Banach space with the Pelczynski's Property (V) has the (RDP) property and every Banach space without copy of l_1 has the (RDP) property. In [27] the authors studied the presence of the (RDP) property in the spaces $C(K, E)$ of all continuous functions on a compact space K with value in a Banach space E . In [13] the second author gave a characterization of the (RDP) property. In [13], [16] and in [5] it has been proved that property (V) and property (RDP) lift from certain Banach spaces X and Y to their projective or injective tensor product.

A Banach space X has the Gelfand Phillips property if every its limited set is relatively compact. Banach spaces having the Gelfand Phillips property are, among the others, separable Banach spaces, Schur spaces, separably complemented spaces, reflexive Banach spaces, spaces with weak* sequentially compact dual balls and spaces $C(K)$, where K is both compact and sequentially compact. It is easy to see that the Gelfand Phillips property, as the (DPrpc), is inherited by closed subspaces. We will use the notations $E \otimes_{\pi} F$ and $E \otimes_{\epsilon} F$, respectively, for the complete projective and the completed injective tensor product of E and F (see [8] for the needed theory of tensor products).

To obtain our first result we need the following well known

Theorem 2. [23] *Let X be a Banach space without a copy of l_1 . Let $M \subset K(X, Y)$ such that*

- 1) *for every $x \in X$, $M(x) = \{T(x) : T \in M\}$ is relatively compact in Y*
 - 2) *M is weakly-norm sequentially equicontinuous that is $\lim_n \sup_{T \in M} \|T(x_n)\| = 0$ for every weakly null sequence $(x_n) \subset X$*
- then M is relatively compact.*

Theorem 3. *Let X and Y be Banach spaces. If X^* and Y have the (DPrpc) and, for every $T \in L(X, Y^{**})$, for every weakly null sequence $(x_n) \subset X$, the sequence $(T(x_n))$ is an L-set, then $K(X, Y)$ has the (DPrpc).*

PROOF. Let $M \subset K(X, Y)$ be a Dunford Pettis set. Then, for every $x \in X$, $M(x)$ is a Dunford Pettis in Y and, since Y has the (DPrpc) property, it is also a relatively compact set. So condition (1) of Mayoral's Theorem is satisfied. Suppose that condition (2) is not verified. Then there are a positive number ϵ , a weakly null sequence $(x_n) \subset X$ and a sequence $(T_n) \subset M$ such that

$$\|T_n(x_n)\| > \epsilon \quad \forall n \in \mathbb{N}.$$

For every $y^* \in Y^*$, the set $\{T_n^* y^* : n \in \mathbb{N}\}$ is also a Dunford Pettis subset of X^* and then it is relatively compact. Therefore, since (x_n) is weakly null, for every $y^* \in Y^*$, it follows that

$$\langle T_n(x_n), y^* \rangle = \langle T_n^*(y^*), x_n \rangle \rightarrow 0.$$

So the sequence $(T_n(x_n))$ is weakly null. Now we prove that $(T_n(x_n))$ is a Dunford Pettis set. Let (y_n^*) be a weakly null sequence in Y^* . The sequence $(x_n \otimes y_n^*)$ is weakly convergent in $X \otimes_{\pi} Y^*$. Indeed, let $H \in (X \otimes_{\pi} Y^*)^* = L(X, Y^{**})$. Since $(H(x_n))$ is an L-set in Y^{**} and (y_n^*) is weakly null in Y^* , then

$$H(x_n \otimes y_n^*) = \langle H(x_n), y_n^* \rangle \rightarrow 0.$$

Since $X \otimes_{\pi} Y^*$ embeds into $(K(X, Y))^*$, it follows that $(x_n \otimes y_n^*)$ is weakly convergent also in the space $(K(X, Y))^*$. Then, since (T_n) is a Dunford Pettis set, it must be

$$\lim_n \langle T_n(x_n), y_n^* \rangle = \lim_n \langle T_n, x_n \otimes_{\pi} y_n^* \rangle = 0$$

So we have proved that $(T_n(x_n))$ is a Dunford Pettis set and then, again by the (DPrpc) property of Y , it must be a compact set. Since it is a weakly null sequence, it follows that there is a norm null subsequence and it is a contradiction. \square

EXAMPLE 4. *The hypothesis that, for every $T \in L(X, Y^{**})$, for every weakly null sequence $(x_n) \subset X$, the sequence $(T(x_n))$ is an L -set, is more general than the hypothesis that $L(X, Y^{**}) = K(X, Y^{**})$. As we shall prove in the following Proposition, the condition $L(X, Y^{**}) = K(X, Y^{**})$ is equivalent to the hypothesis in Ghenciu and Lewis's Theorem [19]. In the following example we consider two Banach spaces X and Y without the Dunford Pettis property since if one of them has the Dunford Pettis property, then the thesis of Theorem 3 follows immediately and it is contained in [19]. Therefore, let us consider the spaces $X = l_p \times c_0$ and $Y = E \times l_q$ with $1 < q < p < \infty$ where E is the first Bourgain Delbaen space. E is an \mathcal{L}_∞ space with the Schur property so it has the (DPrp). As an \mathcal{L}_∞ space, its second dual space embeds into a $C(K)$ space so it has property (V) of Pelczynski. It follows that $E^* = M([0, 1])$ has property (V^*) and then, since it is not reflexive, it must contain a copy of l_1 . By [20], see also [12], E^* must contain a complemented copy of l_1 and therefore E^{**} must contain a copy of c_0 [10][Th. V.10]. This allows us to say that $L(F, E^{**}) \neq K(F, E^{**})$ for every Banach space F , in particular*

$$L(l_p, E^{**}) \neq K(l_p, E^{**}) \quad (1).$$

Now let $T : X \rightarrow Y^{**} = E^{**} \times l_q$ be a linear continuous operator and let

$$p_1 : Y^{**} \rightarrow E^{**} \quad p_2 : Y^{**} \rightarrow l_q$$

be the natural projections. Define $T_1 : l_p \rightarrow E^{**}$ and $T_2 : l_p \rightarrow l_q$ by the laws

$$T_1(a) = p_1(T(a, 0)) \quad \forall a \in l_p$$

$$T_2(a) = p_2(T(a, 0)) \quad \forall a \in l_p$$

Obviously for every $x = (a, b) \in l_p \times c_0$ one has

$$\begin{aligned} T(a, b) &= T(a, 0) + T(0, b) = (p_1(T(a, 0)), p_2(T(a, 0))) + T(0, b) = \\ &= (T_1(a), T_2(a)) + T(0, b) \end{aligned}$$

Let $(x_n) = (a_n, b_n)$ be a weakly null sequence in X . Since c_0 has the Dunford Pettis property, the sequence $(T(0, b_n))$ is a Dunford Pettis set. Since E^{**} has the Dunford Pettis property, then $(T_1(a_n))$ is a Dunford Pettis set. Moreover, by Pitt's Theorem, T_2 is a compact operator, so $(T_2(a_n))$ is norm null and then it is also a Dunford Pettis set. It follows that $(T(x_n)) = (T(a_n, b_n))$ is a Dunford Pettis set and then an L -set. On the other hand, from condition (1) we obtain that $L(X, Y^{**}) \neq K(X, Y^{**})$. \square

The following propositions give some relations about the hypotheses of Theorem 3 and the hypotheses of Ghenciu and Lewis's Theorem ([19][Th. 3.8]).

Proposition 5. *If X^* has the (DPrp) and Y^* does not contain a copy of l_1 , then the following assertions are equivalent*

- a) for every $T \in L(X, Y^{**})$ and for every weakly null sequence $(x_n) \subset X$, $(T(x_n))$ is an L -set
- b) $L(X, Y^{**}) = K(X, Y^{**})$
- c) $L(Y^*, X^*) = K(Y^*, X^*)$

PROOF. $a) \rightarrow b)$ Let $T \in L(X, Y^{**})$. Let $(x_n) \subset X$ be a weakly null sequence, then, by the hypothesis, $(T(x_n))$ is an L-set in Y^{**} . Since Y^* does not contain l_1 , then $(T(x_n))$ is a compact set [11]. Every subsequence has a norm null subsequence so $(T(x_n))$ is norm null. Then T is completely continuous and we are done since X does not contain l_1 .

$b) \rightarrow c)$ It is enough to observe that if $T \in L(Y^*, X^*)$, then $T = (T|_X^*)_{|Y^*}^* : Y^* \rightarrow X^*$ and $T|_X^*$ is compact.

$c) \rightarrow a)$ Conversely, let $T \in L(X, Y^{**})$, and suppose that there is a weakly null sequence $(x_n) \subset X$ such that $(T(x_n))$ is not an L-set. So there are an $\epsilon > 0$, and a weakly null sequence $(y_n^*) \subset Y^*$ such that (passing to a subsequence, if necessary)

$$|\langle T(x_n), y_n^* \rangle| > \epsilon \quad \forall n \in \mathbb{N}$$

$$|\langle T|_{Y^*}^*(y_n^*), x_n \rangle| > \epsilon \quad \forall n \in \mathbb{N}$$

Since $T|_{Y^*}^*$ is compact, then there is a subsequence $(y_{n_k}^*)_k$ such that $(T^*(y_{n_k}^*))$ is norm null and we have a contradiction. So $(T(x_n))$ is an L-set. \square

REMARK 6. *The hypotheses that Y^* does not contain l_1 and that X^* has the (DPrpc) need only in the implication $a) \rightarrow b)$.* \square

Proposition 7. *If $L(Y^*, X^*) = K(Y^*, X^*)$, then at least one of the following conditions holds:*

- a) X^* has (DPrpc)
- b) Y^* does not contain l_1 .

PROOF. Suppose that X^* does not have the (DPrpc). So X contains a copy of l_1 , then X^* does not have the Compact Range Property (in short (CRP)) We refer the reader to [24] and [33] for the definition of the *CRP*. Let T be an integral operator from Y^* into X^* . By the hypothesis, in particular, it is compact. By [6][Th 4.9], Y^* does not contain l_1 . \square

Corollary 8. *If X^* has the Schur property then $(X \otimes_\epsilon X)^*$ and $(X \otimes_\pi X)^*$ have the (DPrpc).*

PROOF. Since X^* has the Schur property, it has the (DPrpc) property. Moreover, since X has the Dunford Pettis property, every linear operator from X into X^{***} sends weakly null sequences into L-sets. Then $K(X, X^*)$ has the (DPrpc). Since X^* is Schur then $(X \otimes_\pi X)^* = L(X, X^*) = K(X, X^*)$ Moreover since X does not contain a copy of l_1 , then $(X \otimes_\epsilon X)^* = I(X, X^*) \subset K(X, X^*)$ where $I(X, X^*)$ is the space of all integral operators from X into X^* . Then $(X \otimes_\epsilon X)^*$ and $(X \otimes_\pi X)^*$ have the (DPrpc) and we are done. \square

REMARK 9. *In particular, Corollary 8 gives a simpler proof of Leung's Theorem, (see [22]), that states that, if H is the Hagler space, then $H \otimes_\epsilon H$ cannot contain a copy of l_1 .* \square

Recently the second author obtained the following result on the compactness of a subset in $\mathcal{K}(X, Y)$:

Theorem 10. [17] *Let X be a Banach space such that X^* has the Gelfand Phillips property. Let $M \subset K(X, Y)$ such that*

- 1) *for every $x \in X$, $M(x) = \{T(x) : T \in M\}$ is relatively compact in Y*
 - 2) *for every weakly star null sequence $(x_n^{**}) \subset X^{**}$, then $(T^{**}(x_n^{**}))$ is norm null uniformly with respect $T \in H$.*
- then M is relatively compact.* □

In [28] the authors, using Theorem 2, gave a sufficient condition in order that a closed subspace of the space $K(X, Y)$ has the Gelfand Phillips property. Using Theorem 10, we can state the following

Theorem 11. *If X^* has the Gelfand Phillips property, if M is a closed subspace of $K(X, Y)$ and, for every $x^{**} \in X^{**}$, the evaluation map $\phi_{x^{**}}$ on M is limited completely continuous, then M has the Gelfand Phillips property.*

PROOF. Let $H \subset M$ be a limited set. By Theorem 10, we have to prove that

- 1) for every $x \in X$, $H(x) = \{T(x) : T \in H\}$ is relatively compact in Y ,
- 2) for every weak* null sequence $(x_n^{**}) \subset X^{**}$, then $(T^{**}(x_n^{**}))$ is norm null uniformly with respect $T \in H$. Condition (1) follows from the hypothesis that, in particular, for every $x \in X$, the map ϕ_x is limited completely continuous. Now, let $(x_n^{**}) \subset X^{**}$ be a weak star null sequence. For every $T \in H$, the adjoint operator T^* is compact, then $T^*(B_{Y^*})$ is relatively compact and therefore it is a limited set. Hence T^* is a limited operator and then it is easy to see that T^{**} is weak*-norm continuous. It follows that $(T^{**}(x_n^{**}))$ is norm null, so $(\phi_{x_n^{**}})$ is a pointwise norm null sequence of linear continuous operators. Then it converges uniformly on the limited sets [32]. It follows that

$$\limsup_n \sup_{T \in H} \|\phi_{x_n^{**}}(T)\| = 0$$

and we are done. □

Corollary 12. *Y has the Gelfand Phillips property if and only if, for every Banach space X such that X^* has the Gelfand Phillips property, then every evaluation map $\phi_{x^{**}} : K(X, Y) \rightarrow Y$ is limited completely continuous.* □

In [31] the authors introduced the so called Right topology on a Banach space X . It is the restriction of the Mackey topology $\tau(X^{**}, X)$ to X and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact sets of X^* . An operator $T : X \rightarrow Y$ is said pseudo weakly compact if it takes Right null sequences of X into norm null sequences of Y . In [7] it was proved that a sequence (x_n) in a Banach space X is Right null if and only if it is a Dunford Pettis set and weakly null.

Definition 13. [31] *A Banach space X is said sequentially Right if, for any Banach space Y , every pseudo weakly compact operator $T : X \rightarrow Y$ is weakly compact.* □

Definition 14. [25] *A subset K of X^* is said a Right set if for every Right null sequence $(x_n) \subset X$ one has*

$$\limsup_n \sup_{x^* \in K} |x^*(x_n)| = 0$$

□

Theorem 15. [25] *X is sequentially Right if and only if every Right set in X^* is relatively weakly compact.* □

Proposition 16. *The following assertions are equivalent*

- a) every bounded set in X^* is a Right set
- b) every Right null sequence is norm null
- c) X has the (DPrpc).

PROOF. a) \rightarrow b) Suppose that there is a Right null sequence (x_n) that is not norm null. So there is an $\epsilon > 0$ and a subsequence (x_{n_k}) such that

$$\|x_{n_k}\| > \epsilon \quad \forall k \in \mathbb{N}$$

Then, for every $k \in \mathbb{N}$, there is $x_k^* \in B_{X^*}$ such that

$$|x_k^*(x_{n_k})| > \epsilon$$

Since (x_k^*) is a Right set we have a contradiction.

b) \rightarrow c) Suppose that A is a Dunford Pettis set that is not relatively compact. So there is an $\epsilon > 0$ and a subsequence $(x_n) \in A$ such that $\|x_n - x_m\| > \epsilon$ for every $n, m \in \mathbb{N}$ with $n \neq m$. Since A is Dunford Pettis set, it is also conditionally weakly compact. So there is a weakly Cauchy subsequence (x_{n_k}) . Hence the sequence $(x_{n_k} - x_{n_{k+1}})$ is weakly null. Since it is contained in $A - A$, it is also a Dunford Pettis sequence then it is Right null. By the hypothesis it is norm null and it leads to a contradiction.

c) \rightarrow a) Let K be a bounded set in X^* and let (x_n) be a Right null sequence. Since X has the (DPrpc), (x_n) is a compact set therefore it is norm null. It follows that

$$\limsup_n \sup_{x^* \in K} |x^*(x_n)| \leq \sup_{x^* \in K} \|x^*\| \|x_n\| \rightarrow 0$$

and it proves that K is a Right set. □

Corollary 17. *A Banach space X is reflexive if and only if it has the (DPrpc) and it is sequentially Right.*

PROOF. If X is reflexive, it has Property (V) of Pelczynski, then it is also sequential Right, moreover, since X^* does not contain l_1 , then X^{**} has the (DPrpc) and then X has the same property. The converse follows from Theorem 15 and Proposition 16. □

In the following Theorem we give a sufficient condition in order that the projective tensor product of two sequentially Right spaces enjoys the same property.

Theorem 18. *If X and Y are sequentially Right and $L(X, Y^*) = K(X, Y^*)$ then $X \otimes_\pi Y$ is sequentially Right.*

PROOF. Let $H \subset (X \otimes_\pi Y)^* = L(X, Y^*) = K(X, Y^*)$ be a Right set. We prove that H is relatively weakly compact. Let (T_n) be a sequence in H . Using the compactness of each T_n , standard arguments allow us to suppose that there is a weak* dense countable set $M \subset Y^{**}$. For every $y^{**} \in M$ we define $\phi_{y^{**}} : K(X, Y^*) \rightarrow X^*$ by putting $\phi_{y^{**}}(T) = T^*y^{**}$. Suppose that $\{\phi_{y^{**}}(T_n) : n \in \mathbb{N}\}$ is not a Right set in X^* .

Then there are $\epsilon > 0$ and a Right null sequence $(x_n) \subset X$ such that (passing to a subsequence if necessary)

$$|\langle y^{**}, T_n x_n \rangle| > \epsilon \quad \forall n \in \mathbb{N}.$$

So $(T_n(x_n))$ is not weakly null. We claim that it cannot be weakly convergent to an element $y_0^* \neq 0$, too. By contradiction, assume that $(T_n(x_n))$ weakly converges to some $y_0^* \neq 0$. If $y_0 \in Y$ and $y_0^*(y_0) \neq 0$, then, in particular,

$$\lim_n \langle y_0, T_n(x_n) \rangle = y_0^*(y_0)$$

Now we observe that $(x_n \otimes y_0)$ is Right null. Indeed, if $A \in (X \otimes_\pi Y)^* = L(X, Y^*) = K(X, Y^*)$, then

$$|(A(x_n \otimes y_0))| \leq \|A(x_n)\| \|y_0\| \rightarrow 0;$$

so $(x_n \otimes y_0)$ is weakly null. Moreover, if (A_n) is a weakly null sequence in $(X \otimes_\pi Y)^* = L(X, Y^*) = K(X, Y^*)$, then by the famous Kalton's test for weak convergence in spaces of compact operators ([26]), it follows that

$$\lim_n \langle x^{**}, A_n^* y_0 \rangle = 0 \quad \forall x^{**} \in X^{**}$$

that is the sequence $(A_n^*(y_0))$ is weakly null in X^* . Since (x_n) is a Dunford Pettis set, it must be

$$\lim_n \langle x_n, A_n^* y_0 \rangle = 0$$

that is

$$\lim_n A_n(x_n \otimes y_0) = 0$$

So, we can deduce that $(x_n \otimes y_0)$ is Dunford Pettis and then Right null. Since (T_n) is a Right set, we have reached the sought-for contradiction. It follows that $(T_n(x_n))$ is not a relatively weakly compact sequence so, since Y is sequential Right, it is not a Right set. Then there are a Right null sequence (y_n) and $\sigma > 0$ such that

$$|\langle T_n(x_n), y_n \rangle| > \sigma.$$

But the sequence $(x_n \otimes y_n)$ is a Right set ([1]) and we have a contradiction. So we have proved that $\{\phi_{y^{**}}(T_n) : n \in \mathbb{N}\}$ is a Right set in X^* . Since X is sequentially Right, $(\phi_{y^{**}}(T_n))$ has a weakly convergent subsequence. By passing to a subsequence, by countability of M , we can say that, for every $y^{**} \in M$, the sequence $(\phi_{y^{**}}(T_n))$ is weakly convergent. Now, given $x^{**} \in X^{**}$, define

$$\phi_{x^{**}} : K(X, Y^*) \rightarrow Y^*$$

by putting

$$\phi_{x^{**}}(T) = T^{**}(x^{**}) \quad \forall T \in K(X, Y^*).$$

With an argument similar to the one above, one can prove that $\{\phi_{x^{**}}(T_n) : n \in \mathbb{N}\}$ is a Right set and then it is relatively weakly compact. It easy to prove that, if (T_{n_r}) and (T_{n_k}) are two subsequences of (T_n) , then $(\phi_{x^{**}}(T_{n_r}))$ and $(\phi_{x^{**}}(T_{n_k}))$ are weakly convergent to the same element. Therefore, for every subsequence $(\phi_{x^{**}}(T_{n_r}))$, repeating the above proof, there is a subsequence weakly convergent to the same element. It follows that $(\phi_{x^{**}}(T_n))$ is weakly convergent. We can define

$$S : X^{**} \rightarrow Y^*$$

by the law

$$S(x^{**}) := \text{weak} - \lim_n T_n^{**}(x^{**})$$

and

$$B : Y^{**} \rightarrow X^*$$

by the law

$$B(y^{**}) := \text{weak} - \lim_n T_n^*(y^{**})$$

It follows

$$\begin{aligned} \langle x^{**}, B(y^{**}) \rangle &= \lim_n \langle x^{**}, T_n^*(y^{**}) \rangle = \\ &= \lim_n \langle T_n^{**} x^{**}, y^{**} \rangle = \\ &= \langle S(x^{**}), (y^{**}) \rangle = \\ &= \langle x^{**}, S^*(y^{**}) \rangle \end{aligned}$$

Hence,

$$B = S^*.$$

It follows easily that B is weak*-weak continuous, so it is a conjugate operator. This means that there is $T : X \rightarrow Y^*$ such that $B = T^*$. By the hypothesis, T is a compact operator. Let $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ then

$$\begin{aligned} \lim_n \langle y^{**}, T_n^{**}(x^{**}) \rangle &= \langle y^{**}, S(x^{**}) \rangle = \\ &= \langle S^*(y^{**}), x^{**} \rangle = \\ &= \langle B(y^{**}), x^{**} \rangle = \\ &= \langle T^*(y^{**}), x^{**} \rangle = \\ &= \langle y^{**}, T^{**}(x^{**}) \rangle \end{aligned}$$

that is (T_n) is weakly convergent to T , always thanks to Kalton's test ([26]). \square

Corollary 19. $l_p \otimes_\pi Y$ ($p > 1$), where Y is the second Bourgain Delbaen space, and $l_p \otimes_\pi c_0$ have the sequential Right property.

PROOF. For every $p > 1$, the space l_p has the Pelczynski's Property (V) so it is also sequentially Right. The second Bourgain Delbaen space is also sequentially Right [25]. Moreover suppose $1 < p < +\infty$, then every operator $T : l_p \rightarrow Y^*$ is weakly compact. Since Y^* is isomorphic to l_1 it is also compact. If $p = +\infty$ every linear operator $T : l_\infty \rightarrow Y^*$ is absolutely summing since it is defined on an \mathcal{L}_∞ space and has its values in a space with cotype two. Then, as before, it is also a compact operator. The same holds if we replace Y with c_0 . So the hypotheses of Theorem 18 are satisfied. \square

REMARK 20. Since the second Bourgain Delbaen space does not have the Pelczynski property (V) the result in Corollary 19 cannot be deduced from Emmanuele and Hensgen's Theorems ([16]) about the property (V) in the projective tensor products. On the other hand from Emmanuele's Theorem [13] about the property (RDP) we can obtain then $l_p \otimes_\pi Y$ ($p > 1$), where Y is the second Bourgain Delbaen space has the (RDP) property, but the sequential Right property in a Banach space without

the Dunford Pettis property is, in general, a stronger property than the Reciprocal Dunford Pettis property. \square

Corollary 21. *If X and Y are sequentially Right and at least one of X^* or Y^* is a Schur space, then $X \otimes_{\pi} Y$ is sequentially Right.*

PROOF. If Y^* is a Schur space, then every $T \in L(X, Y^*)$ sends Right null sequences of X in norm null sequences of Y^* , so it is pseudo weakly compact. Since X is sequentially Right, every $T \in L(X, Y^*)$ is weakly compact and then again, by the Schur property, it is compact. So we can apply Theorem 18. If X^* is a Schur space, then, with the same arguments, it follows that $Y \otimes_{\pi} X$ is sequentially Right and therefore we are done. \square

Theorem 22. *If X and Y have the Dunford Pettis property the following assertions are equivalent*

- a) X and Y are sequentially Right and at least one of them does not contain l_1
- b) $X \otimes_{\pi} Y$ is sequentially Right and $L(X, Y^*) = K(X, Y^*)$.

PROOF.

a) \rightarrow b) It follows from the proof of Corollary 21 since one of the space X^* or Y^* is a Schur space.

b) \rightarrow a) Obviously X and Y are sequentially Right. Since $X \otimes_{\pi} Y$ has the Reciprocal Dunford Pettis property and $L(X, Y^*) = K(X, Y^*)$, then we can apply Theorem 8 in [13]. \square

Theorem 23. *If X is an \mathcal{L}_{∞} space and X^*, Y are sequentially Right, then $K(X, Y)$ is sequentially Right too.*

PROOF. Let $H \subset (K(X, Y))^*$ be a Right set. Since $K(X, Y) = X^* \otimes_{\epsilon} Y$ it follows that H is a Right subset of $(X^* \otimes_{\epsilon} Y)^*$. Since X^* is an \mathcal{L}_{∞} space, there is an isomorphism S from $(X^* \otimes_{\epsilon} Y)^*$ into $C(K, Y)^*$ where $K = B_{X^{***}}$ endowed with the weak* topology ([4]). S is obtained as the restriction of the adjoint of the projection $P : C(K, Y) \rightarrow X^{***} \otimes_{\epsilon} Y$ to the subspace $(X^* \otimes_{\epsilon} Y)^*$ of $(X^{***} \otimes_{\epsilon} Y)^*$. It is easy to prove that $S(H)$ is a Right set in $C(K, Y)^*$. Since Y is sequentially Right, $C(K, Y)$ is sequentially Right ([25][Th 4.2]) then $S(H)$ is weakly compact. Since S is an isomorphism, then H is weakly compact too. \square

Definition 24. [29] *A subset A of a dual space X^* is called an L -limited set if every weak null and limited sequence (x_n) in X converges uniformly on A . A Banach space X has the L -limited property if every L -limited set in X^* is relatively weakly compact.* \square

Theorem 25. *If X and Y have the L -limited property and $L(X, Y^*) = K(X, Y^*)$ then $X \otimes_{\pi} Y$ has the same property.*

PROOF. It follows adapting the proof of Theorem 18 \square

Proposition 26. *A Banach space X has the L -limited property if and only if it is sequentially Right and it has the Grothendieck property.*

PROOF. Suppose that X has the L-limited property. Let $T : X \rightarrow Y$ be a pseudo weakly compact operator. Let (x_n) be a limited and weakly null sequence then it is also a Right null sequence. So $(T(x_n))$ is norm null. Then T is limited completely continuous. Hence, by [29], it is weakly compact and we are done. Moreover by [29][Theorem 2.10], X has the Grothendieck property. Conversely, suppose that X is sequentially Right and that it has the Grothendieck property. Hence every Right null sequence it is also limited and weakly null, then every limited completely continuous operator is also a pseudo weakly compact operator. Since X is sequentially right, it is a weakly compact operator, therefore X has the L-limited property. \square

As for the Property (V) of Pelczynski and the (RDP) property, we can introduce the so called (SR^*) property that is a dual property with respect the sequential Right property.

Definition 27. *A bounded subset K of a Banach space X is said Right* set if for every Right null sequence (x_n^*) in X^* it follows*

$$\limsup_n \sup_{x \in K} |x_n^*(x)| = 0$$

A Banach space X has the (SR^) property if every Right* set is relatively weakly compact.* \square

As for the well known properties (V) and (RDP), if E has the sequential Right property, then its dual has the (SR^*) property.

Proposition 28. *If every conjugate pseudo weakly compact operator $T^* : E^* \rightarrow F^*$ is weakly compact, then E has the (SR^*) property.*

PROOF.

Let K be a Right* set. Let $(x_n)_n \subset K$. Define

$$S : l_1 \rightarrow E$$

by the law

$$S(y) := \sum_{n=1}^{\infty} y_n x_n \quad \forall y = (y_n) \in l_1.$$

It is trivial to prove that S is a linear continuous operator. Observe that

$$\begin{aligned} \langle S^* x^*, y \rangle &= \langle x^*, S(y) \rangle = \\ &= \langle x^*, \sum_{n=1}^{\infty} y_n x_n \rangle = \\ &= \langle y, (x^*(x_n))_n \rangle \end{aligned}$$

that is

$$S^*(x^*) = ((x^*(x_n))_n)$$

Let $(x_n^*) \subset X^*$ be a Right null sequence and suppose that $(S^*(x_n^*))$ is not norm null. Then, we can suppose that there is an $\epsilon > 0$ such that

$$\|S^*(x_n^*)\| > \epsilon$$

that is

$$\sup_m |x_n^*(x_m)| > \epsilon \quad \forall n \in \mathbb{N}$$

For every $n \in \mathbb{N}$ there is x_{m_n} such that

$$|x_n^*(x_{m_n})| > \epsilon$$

Since (x_{m_n}) is a Right* set and (x_n^*) is Right null we have a contradiction. So we have proved that S^* is pseudo weakly compact. By the hypothesis it is also weakly compact. Then $(x_n) = (S(e_n))$ is weakly convergent and we are done. \square

Since every pseudo weakly compact operator is an unconditional operator and since every Dunford Pettis operator is pseudo weakly compact, using well known characterizations of properties (V^*) and (RDP^*) (see [14]), it follows that

$$(V^*) \rightarrow (SR^*) \rightarrow (RDP^*).$$

The space $E = (\sum l_\infty^n)_1$ has property (V^*) then it has the (SR^*) property but E^{**} contains l_∞ then it is not weakly sequentially complete. Hence E^* does not have the (SR) property.

We end by considering two more isomorphic properties

Definition 29. *A Banach space X has the Bourgain Diestel Property, (in short (BD) property, [3]) if every its limited subset is relatively weakly compact.*

A Banach space X has the (RDP^) property if every its Dunford Pettis subset is relatively weakly compact.* \square

Theorem 30. *If X and Y have the (BD) property (respectively the (RDP^*) property) and $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, then $K_{w^*}(X^*, Y)$ has the same property.*

PROOF. The two proofs are similar so we give just the one about (RDP^*) property. So let (h_n) be a Dunford Pettis sequence; hence (h_n) is a weakly conditionally compact set and we can suppose that it is a weak Cauchy sequence. So for all $x^* \in E^*$ the sequence $(h_n(x^*))$ is a Dunford Pettis sequence that is also weakly Cauchy. Since F has the (RDP^*) property, we can conclude that there is the weak limit in F . Define $h : E^* \rightarrow F$ by putting

$$h(x^*) := w - \lim_n h_n(x^*).$$

It is trivial to prove that $h \in L(E^*, F)$. Similarly we are allowed to define $k : F^* \rightarrow E$ by putting

$$k(y^*) := w - \lim_n h_n^*(y^*)$$

It is easy to prove that $h^* = k$ and that h is $w^* - w$ continuous. Then $h \in L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$. Now we prove that (h_n) is weakly convergent to h . By the Rainwater's Theorem, we have to prove that for each $\phi^* \in \text{ext}B_{K_{w^*}(X^*, Y)}$, then $\phi^*(h_n - h) \rightarrow 0$. By Ruess and Stegall Theorem there are x^* and y^* such that $\phi^* = x^* \otimes y^*$. By the definition of h ,

$$h_n(x^* \otimes y^*) = h_n(x^*)(y^*) \rightarrow h(x^*)(y^*)$$

and we are done. \square

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SOME ISOMORPHIC PROPERTIES IN $K(X, Y)$ AND IN PROJECTIVE TENSOR PRODUCTS¹⁵

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