JOURNAL OF FUNCTIONAL ANALYSIS 99, 125–130 (1991)

# Remarks on the Uncomplemented Subspace $W(E, F)$*

G. EMMANUELE

*Department of Mathematics, University of Catania, 95125 Catania, Italy*

*Communicated by the Editors*

Received September 27, 1989; revised February 24, 1990

We give some theorems showing that $W(E, F)$ is uncomplemented in $L(E, F)$. All of them (but Theorem 5) are proved by producing a complemented copy of $c_0$ inside of $W(E, F)$ and showing that this contradicts a possible complementation of $W(E, F)$ in $L(E, F)$. 

Let $E$, $F$ be two Banach spaces. By $L(E, F)$ (resp. $W(E, F)$) we denote the Banach space of all bounded, linear (resp. bounded, linear, weakly compact) operators from $E$ into F. Many papers (see [6–8] and their references) have been devoted to the following problem: when is $K(E, F) = \{\text{compact operators from } E \text{ into } F\}$ uncomplemented in $L(E, F)$? No results exist about the position of $W(E, F)$ inside of $L(E, F)$, as far as we know. The present note deals with this last question; more precisely, we prove that in special cases $W(E, F)$ is not complemented in $L(E, F)$. We note that the presence in $W(E, F)$ of a copy of $c_0$ does not avoid that $W(E, F) = L(E, F)$, as it happened in the case of $K(E, F)$ in special settings (see [6–8] and their references); for instance $W(l_2, l_2) = L(l_2, l_2)$ and $W(l_2, l_\infty) = L(l_2, l_\infty)$ both contain a copy of $c_0$. Such a copy is not complemented in them, because they are dual Banach spaces.

Motivated by these remarks, we try to construct special complemented copies of $c_0$ inside of $W(E, F)$, and then prove that this fact is not compatible with a possible complementation of $W(E, F)$ inside of $L(E, F)$. In all of our results we shall assume that one (or more) of $E$, $E^*$, $F$, $F^*$ contains a copy of $c_0$, plus some other assumptions; after all, the above examples show that some strong hypothesis must be considered.

The technique we use in the case of $W(E, F)$ is, sometimes, useful even for detecting the possible complementation of $K(E, F)$ in $L(E, F)$, as we prove at the end of the paper giving a different proof of the following result

* Work performed under the auspices of G.N.A.F.A. of C.N.R. and partially supported by M.U.R.S.T. of Italy.



0022-1236/91 $3.00

due to Feder: "$K(C(S), L^1)$ is not complemented in $L(C(S), L^1)$, if $S$ is not dispersed" [6], when answering a question put by Johnson in [7].

Before giving our results we need two definitions and a theorem about complemented copies of $c_0$ in Banach spaces.

DEFINITION 1 [2]. A (bounded) subset $X$ of a Banach space $E$ is called a *limited* subset if for each weak* null sequence $(x_n^*) \subset E^*$ one has

$$\lim_n \sup_X |x_n^*(x)| = 0.$$

DEFINITION 2 [2]. A Banach space $E$ is said to have the *Gelfand–Phillips property* if any of its limited subsets is relatively compact.

THEOREM 1 [5, 11]. *Let $(x_n)$ be a copy of the unit vector basis of $c_0$ in E. If $\overline{\operatorname{span}}(x_n)$ is complemented in E, then $(x_n)$ is not a limited subset. Conversely if $(x_n)$ is not limited, then a suitable subsequence spans a complemented copy of $c_0$ in E.*

Now, we are ready to prove our theorems; in the first one, we even require that $E$ has the Dunford–Pettis property but not the Schur property (see [2] for these well-known definitions).

THEOREM 2. *Assume E is a Banach space enjoying the Dunford–Pettis property, the Gelfand–Phillips property, but not the Schur property. If F contains a copy of $c_0$, then $W(E, F)$ is uncomplemented in $L(E, F)$.*

*Proof.* Let $(x_n) \subset E$ be a weak null sequence of norm one elements. Since $E$ has the Gelfand–Phillips property, $(x_n)$ is not limited. There is a weak* null sequence $(x_n^*) \subset E^*$ with $\inf_n |x_n^*(x_n)| > 0$. By virtue of a result in [11] we can assume (and we do) that $x_m^*(x_n) = \delta_{mn}$. Let $(y_n) \subset F$ be a copy of the unit vector basis of $c_0$ and $(y_n^*)$ a sequence of biorthogonal coefficients. We consider the sequences $(x_n^* \otimes y_n) \subset W(E, F)$ and $(x_n \otimes y_n^*) \subset W^*(E, F)$. Clearly, one has $(x_m^* \otimes y_m)(x_n \otimes y_n^*) = \delta_{mn}$ and that $(x_n^* \otimes y_n)$ is a copy of the unit vector basis of $c_0$. Now, we show that $x_n \otimes y_n^* \xrightarrow{w^*} \vartheta$. Let $T \in W(E, F)$. We have

$$|T(x_n \otimes y_n^*)| \leqslant \|y_n^*\| \; \|T(x_n)\| \leqslant \text{const.} \; \|T(x_n)\| \to 0$$

because $x_n \xrightarrow{w} \vartheta$ and $T$ is a Dunford–Pettis operator, since $T \in W(E, F)$ and $E$ has the Dunford–Pettis property. Thanks to Theorem 1 we have that a subsequence $(x_{k(n)}^* \otimes y_{k(n)})$ of $(x_n^* \otimes y_n)$ spans a complemented copy of $c_0$. Now, define $\psi \colon l_\infty \to L(E, F)$ by putting

$$\psi(\xi)(x) \equiv \sum_{n=1}^{\infty} \xi_n x_{k(n)}^*(x) \, y_{k(n)}, \qquad \xi \in l_\infty, \; x \in E.$$

$\psi$ is well defined since $x^*_{k(n)} \xrightarrow{w^*} \vartheta$, $\xi \in l_\infty$, and $(y_{k(n)})$ is equivalent to the unit vector basis of $c_0$. It is clear that $\psi$ is linear. Furthermore, we have that there is $C_2 > 0$ such that

$$\|\psi(\xi)(x)\| \leqslant C_2 \sup_n |\xi_n x^*_{k(n)}(x)| \leqslant C'_2 \|\xi\|_\infty, \qquad x \in B_E, \xi \in l_\infty,$$

from which follows the boundedness of $\psi$ (we have clearly used the fact that $(y_{k(n)})$ is a copy of the unit vector basis of $c_0$). On the other hand, thanks to the nature of $(y_{k(n)})$, there is $C_1 > 0$ so that

$$\|\psi(\xi)(x)\| \geqslant C_1 \sup_n |\xi_n x^*_{k(n)}(x)| \geqslant C_1 |\xi_h x^*_{k(h)}(x)|, \qquad x \in B_E, h \in N, \xi \in l_\infty$$

and hence

$$\|\psi(\xi)\| = \sup_{B_E} \|\psi(\xi)(x)\| \geqslant C'_1 \|\xi\|_\infty, \qquad \xi \in l_\infty.$$

$\psi$ is then an isomorphism from $l_\infty$ onto a closed subspace $H$ of $L(E, F)$, mapping $c_0$ onto a closed subspace $H_1$ of $H \cap W(E, F)$. If we assume by contradiction that $W(E, F)$ is complemented in $L(E, F)$, it turns out that $H_1$ is complemented in $L(E, F)$ too (remember that $H_1$ was complemented in $W(E, F)$). Hence, we should obtain that $H_1$ is complemented in $H$, i.e., $c_0$ is complemented in $l_\infty$, a fact that is known to be false. The proof is complete.

*Remark* 1. Since $(x^*_n)$ can be chosen in $S_{E^*}$, if $(y_n)$ is an isometric copy of the unit vector basis of $c_0$ (i.e., $c_1 = c_2 = 1$), then $H_1$ is an isometric complemented copy of $c_0$ inside of $W(E, F)$.

In the above Theorem 2 the assumptions on $E$ were considered just in order to guarantee that the considered copy of the unit vector basis of $c_0$ spanned a complemented subspace of $W(E, F)$. Hence, with the same proof, the following result is true (it is similar to Theorem 4 of [7]).

THEOREM 3. *Assume $F$ contains a complemented copy of $c_0$. If $E^*$ contains a weak* null sequence that is not weak null, then $W(E, F)$ is uncomplemented in $L(E, F)$.*

*Proof.* Let $(y_n)$ be a copy of the unit vector basis of $c_0$, spanning a complemented subspace of $F$. Take a weak* null sequence $(x^*_n) \subset E^*$ that is not weak null. We just have to show that a suitable subsequence of $(x^*_n \otimes y_n)$ is a copy of the unit vector basis of $c_0$ spanning a complemented subspace of $W(E, F)$ and then apply the proof of Theorem 2. We observe that $W(E, F)$ is isometrically isomorphic to $L_{W^*}(E^{**}, F)$, i.e., the Banach space of all weak*–weak continuous operators from $E^{**}$ into $F$. Hence $(x_n^* \otimes y_n)$ is a copy of the unit vector basis of $c_0$ even inside of $L_{W^*}(E^{**}, F)$. Let $x^{**}$ be an element of $E^{**}$ such that $\inf_n |x_n^*(x^{**})| > 0$ (otherwise we pass to a subsequence) and let $(y_n^*)$ be a weak* null sequence in $F^*$, with $y_m^*(y_n) = \delta_{mn}$. It is clear that $(x^{**} \otimes y_n^*) \subset (L_{W^*}(E^{**}, F))^*$ and that $\inf_n |(x^{**} \otimes y_n^*)(x_n^* \otimes y_n)| > 0$. Furthermore, if $T \subset L_{W^*}(E^{**}, F)$, one has

$$|T(x^{**} \otimes y_n^*)| = |[T(x^{**})](y_n^*)| \to 0$$

and so $(x^{**} \otimes y_n^*) \xrightarrow{w^*} \vartheta$. Again Theorem 1 gives the existence of a complemented copy of $c_0$ in $(L_{W^*}(E^{**}, F)$ and hence in$)$ $W(E, F)$, spanned by a suitable subsequence of $(x_n^* \otimes y_n)$. We are done.

We observe that $L(l_\infty, c_0) = W(l_\infty, c_0)$, so that the hypothesis on $E^*$ in Theorem 3 cannot be dropped.

COROLLARY 4. *Assume $E$ and $F$ contain a complemented copy of $c_0$. Then $W(E, F)$ is uncomplemented in $L(E, F)$.*

The next theorem has a different nature, because we do not require the presence of a copy of $c_0$ inside of $F$.

THEOREM 5. *Let $F$ contain a complemented copy of $l^1$. Assume $E$ is such that $L(E, l^1) \neq K(E, l^1)$. Then $W(E, F)$ is uncomplemented in $L(E, F)$.*

*Proof.* As can be easily checked $W(E, l^1)$ is complemented in $W(E, F)$. Since $l^1$ has the Schur property, $W(E, l^1) = K(E, l^1)$. Let $T$ denote an element of $L(E, l^1)$ not in $K(E, l^1)$. Using the existence of an unconditional basis in $l^1$ we can easily construct a series $\sum T_n$ in $K(E, l^1)$ such that $\sum T_n(x)$ converges unconditionally to $T(x)$, for all $x \in E$. A result due to Feder [6] gives that $K(E, l^1)$ is uncomplemented in $L(E, l^1)$. Now, assume by contradiction that $W(E, F)$ is complemented in $L(E, F)$; it follows that $W(E, l^1)$ is complemented in $L(E, F)$. On the other hand, we have the following chain of (isometric) embeddings

$$K(E, l^1) \hookrightarrow L(E, l^1) \hookrightarrow L(E, F). \tag{1}$$

By virtue of the previous remarks and (1) alike, $K(E, l^1)$ should be complemented in $L(E, l^1)$, a contradiction concluding the proof.

COROLLARY 6. *Assume $F$ contains a complemented copy of $l^1$ and $E$ has a quotient isomorphic to $l^1$. Then $W(E, F)$ is uncomplemented in $L(E, F)$.*

*Remark* 2. The same proof of Theorem 5 shows that the following variation of that result is true.

THEOREM 5′. *Let $F$ contain a complemented copy of a Banach space $Z$ with the Schur property. If $E$ contains a complemented copy of $l^1$, then $W(E, F)$ is uncomplemented in $L(E, F)$.*

We only remark that, under such hypotheses, $K(E, Z)$ is not complemented in $L(E, Z)$ [8].

Finally, we present the announced new proof of a result by Feder [6] using the same technique of the construction of a complemented copy of $c_0$ inside of $K(E, F)$.

THEOREM 7. *$K(C(S), L^1)$ is not complemented in $L(C(S), L^1)$, if $S$ is a Hausdorff compact space that is not dispersed.*

*Proof.* Since $S$ is not dispersed, $l^1$ embeds into $C(S)$ (see [9]) and, by virtue of a famous theorem of Pelczynski (see [10]), $L^1$ embeds into $C^*(S)$. Now, it is well known that the Rademacher functions span $l^2$ inside of $L^1$. So we are in the following position: there is an isomorphism $R$ (resp. $S$) from $l^2$ into $C^*(S)$ (resp. $L^1$).

We shall show that, if $(e_n)$ is the unit vector basis of $l^2$, then $(R(e_n) \otimes_\varepsilon S(e_n))$ is a copy of the unit vector basis of $c_0$ inside of $K(C(S), L^1)$ (equal to $C^*(S) \otimes_\varepsilon L^1$ because $L^1$ has the approximation property [1]). For brevity we suppose $\|(R(e_n)\| = \|S(e_n)\| = 1$ for all $n \in N$. Since $(R(e_n))$ and $(S(e_n))$ are basic sequences, there are biorthogonal sequences $(x_n^*)$ in $C^{**}(S)$ and $(y_n^*)$ in $L^\infty$. Hence it is clear, from the definition of $\varepsilon$-tensor norm, that, for any choice of $a_i$, $i = 1, ..., n$, we have

$$\max_{1 \leqslant i \leqslant n} |a_i| \leqslant \left\| \sum_{i=1}^{n} a_i (R(e_i) \otimes_\varepsilon S(e_i)) \right\|.$$

On the other hand, $R$ and $S$ induce an operator $R \otimes_\varepsilon S$ from $l^2 \otimes_\varepsilon l^2$ into $C^*(S) \otimes_\varepsilon L^1$ (see [1, p. 229]), mapping $e_n \otimes_\varepsilon e_n$ onto $R(e_n) \otimes_\varepsilon S(e_n)$. So we have

$$\left\| \sum_{i=1}^{n} a_i R(e_i) \otimes_\varepsilon S(e_i) \right\| \leqslant \|R \otimes_\varepsilon S\| \left\| \sum_{i=1}^{n} a_i e_i \otimes_\varepsilon e_i \right\| \leqslant \max_{1 \leqslant i \leqslant n} |a_i| \; \|R \otimes_\varepsilon S\|.$$

Hence $c_0$ embeds isomorphically in $K(C(S), L^1)$. Now, observe that $K(C(S), L^1)$ (equal to $C^*(S) \otimes_\varepsilon L^1$) has the Gelfand–Phillips property [4] and so $c_0$ embeds complementably in $K(C(S), L^1)$, by virtue of a result in [5, and 11]. If we assume, by contradiction, that $K(C(S), L^1)$ is complemented in $L(C(S), L^1)$, being $L^1$ complemented in its bidual $(L^1)^{**}$, $c_0$ should be complemented in $L(C(S), (L^1)^{**})$, the dual space of $C(S) \otimes_\pi L^\infty$, a contradiction that finishes our proof.

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