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INTEGRABLE SOLUTIONS OF A FUNCTIONAL-INTEGRAL EQUATION

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ABSTRACT. We consider a very general functional-integral equation and we prove the existence of integrable solutions of this equation.

In this paper we consider the following functional-integral equation

$$y(t) = f\left(t, r\int_0^1 k(t,s)g(s,y(s))\,ds\right) \quad t \in [0,1] \tag{1}$$

and we prove that, under very general hypotheses, it admits a solution $x \in L^1[0,1]$. We observe that if $f(t,u) = \varphi(t) + u$ we get Hammerstein integral equations (we refer to [**2, 5, 9**] and references therein for papers about existence results concerning this equation as well as for applications of it to other questions), whereas when $g(s,v) = v$ we obtain a functional-integral equation recently studied in [**3**], where the usefulness of it in applications was also pointed out. Our theorem extends all of the known results from [**2, 3, 5, 6, 7** and **9**] because the hypotheses we consider are very general and *natural* in the sense that they are necessary and sufficient conditions for certain (superposition) operators to take $L^1[0,1]$ into itself continuously, see [**8**].

We remark that in the results from [**2, 3, 5**, and **9**] assumptions of monotonicity and coercivity were quite often assumed by the authors, whereas we dispense completely with them; furthermore, in [**3**] Banas and Knap assumed that $k(t,s) \geq 0$ a.e. on $[0,1]^2$; we are able to dispense with this requirement as well as with the following other hypothesis:

There exists $\lambda \in L^1[0,1]$ *such that* $|k(t,s)| \leq \lambda(t)$
t *a.e. on* $[0,1]$, $s \in [0,1]$

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we used in [**6**], or with "regularity" conditions still put on k in the recent [**7**] and in older papers ([see **11**]). All of these improvements are determined by the development of the technique we used in [**7**] that we are able to carry out in the present framework; more precisely in [**7**] we considered an operator A (defined, in the present setting, by

$$(2)\qquad (Ay)(t) = f\left(t, r\int_0^1 k(t,s)g(s,y(s))\,ds\right) \quad t \in [0,1])$$

from a suitable bounded, closed, convex and uniformly integrable (i.e., relatively weakly compact) subset Q of $L^1[0,1]$ into itself and we proved that A is continuous and $A(Q)$ is relatively compact (using heavily the uniform integrability of Q). An attentive inspection of that proof shows that the relative compactness of $A(Q)$ only depends on the uniform integrability of Q, not on the particular form of Q. With this in mind, we spent some time to look for different (and good) kinds of uniformly integrable subsets of $L^1[0,1]$ (for a different result, look at the paper [**7**]), until we realized that there exists a ball B_r of $L^1[0,1]$ containing a nonempty, bounded, closed, convex and uniformly integrable subset Q of B_r that is invariant under the quoted operator A; we do not know the nature of Q, but we know that it exists and this is enough to assert that $A(Q) \subset Q$ is relatively compact, thanks to the technique developed in [**7**]. Hence, the Schauder fixed point theorem applies to get a fixed point of A, i.e., a solution of (1).

The main tools we use are two: a measure of weak noncompactness introduced by De Blasi [**4**] together with a result about its value on a bounded subset of $L^1[0,1]$ [**1**] and a theorem, due to Scorza Dragoni [**10**], about measurable functions of two variables.

Definition 1. [**4**]. Let E be a Banach space and X be a nonempty, bounded subset of E. If B_r denotes the ball centered at θ with radius $r > 0$, we put $\beta(X) = \inf\{r > 0$: there exists a weakly compact subset Y of E with $X \subset Y + B_r\}$.

Theorem 2. [**1**]. *Let X be a nonempty, bounded subset of $L^1[0,1]$, then*

$$\beta(X) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left\{\int_D |x(t)|\,dt : D \subset [0,1], m(D) \le \varepsilon\right\}\right\}\right\}.$$

Theorem 3. [**10**]. *Let $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a function verifying Caratheodory hypotheses, i.e., f is measurable with respect to $t \in [0,1]$ for all $s \in \mathbf{R}$ and continuous in $s \in \mathbf{R}$ for a.a. $t \in [0,1]$. Then given $\varepsilon > 0$ there is a closed subset D_ε of $[0,1]$ with $m(D_\varepsilon^c) < \varepsilon$ and $f|_{D_\varepsilon \times \mathbf{R}}$ continuous.*

Now we are ready to prove our theorem.

Theorem 4. *Let us consider the following hypotheses*

(h_1) *$f : [0,1] \times \mathbf{R} \to \mathbf{R}$ verifies Caratheodory hypotheses and there are $h_1 \in L^1[0,1]$ and $b_1 \geq 0$ such that*

$$|f(t,x)| \leq h_1(t) + b_1|x| \quad t \text{ a.e. in } [0,1],\ x \in \mathbf{R}$$

(h_2) *$k : [0,1] \times [0,1] \to \mathbf{R}$ verifies Caratheodory hypotheses and the linear operator K defined by*

$$(Kz)(t) = \int_0^1 k(t,s)z(s)\,ds \quad t \in [0,1]$$

maps $L^1[0,1]$ into itself (*this fact implies that K is bounded* [**11**]; *let $||K||$ denote the norm of such an operator*)

(h_3) *$g : [0,1] \times \mathbf{R} \to \mathbf{R}$ verifies Caratheodory hypotheses and there are $h_2 \in L^1[0,1]$ and $b_2 \geq 0$ such that*

$$|g(t,x)| \leq h_2(t) + b_2|x| \quad t \text{ a.e. in } [0,1],\ x \in \mathbf{R}.$$

(h_4) *$rb_1b_2||K|| < 1$, $r \geq 0$.*

Then the equation (1) *has a solution $x \in L^1[0,1]$.*

Proof. Let us put $s = (||h_1|| + rb_1||K||\,||h_2||)/(1 - rb_1b_2||K||)$. We first prove that the operator A defined by (2) maps B_s into itself continuously. Let $x \in B_s$. We have

$$\begin{aligned}
\int_0^1 |Ax(t)|\,dt &= \int_0^1 |f(t, r\int_0^1 k(t,s)g(s,x(s))\,ds|dt \\
&\leq \int_0^1 \left\{ h_1(t) + rb_1 \left| \int_0^1 k(t,s)g(s,x(s))\,ds \right| \right\} dt \\
&= ||h_1|| + rb_1 \int_0^1 \left| \int_0^1 k(t,s)g(s,x(s))\,ds \right| dt \\
&\leq ||h_1|| + rb_1 ||K|| \int_0^1 |g(s,x(s))|\,ds \\
&\leq ||h_1|| + rb_1 ||K|| \int_0^1 (h_2(s) + b_2|x(s)|)\,ds \\
&= ||h_1|| + rb_1||K||\,||h_2|| + rb_1 b_2 ||K||\,||x|| \\
&\leq ||h_1|| + rb_1||K||\,||h_2|| + rb_1 b_2 ||K|| s = s.
\end{aligned}$$

The continuity of A is a simple matter to show thanks to our assumptions (h_1), (h_2) and (h_3), so we don't give the details.

Now we show that $\beta(A(X)) \leq rb_1 b_2 ||K|| \beta(X)$ for each subset X of B_s. Toward this aim, we consider two operators F, G defined on $L^1[0,1]$ with values into $L^1[0,1]$ by putting

$$(Fy)(t) = f(t, y(t)) \quad \text{and} \quad (Gy)(t) = g(t, y(t)) \quad t \in [0,1].$$

For a subset $D \subset [0,1]$, we have

$$\begin{aligned}
\int_D |(Fy)(t)|\,dt &\leq \int_D h_1(t)\,dt + b_1 \int_D |y(t)|\,dt \quad y \in X \\
\int_D |(Gy)(t)|\,dt &\leq \int_D h_2(t)\,dt + b_2 \int_D |y(t)|\,dt \quad y \in X.
\end{aligned}$$

Since $\lim_{m(D)\to 0} \int_D h_1(t)\,dt = \lim_{m(D)\to 0} \int_D h_2(t)\,dt = 0$, Theorem 2 allows us to affirm that

$$\beta(F(X)) \leq b_1 \beta(X) \quad \text{and} \quad \beta(G(X)) \leq b_2 \beta(X). \tag{3}$$

Moreover, since K is linear and continuous, it is easy to see that

$$\beta(K(X)) \leq ||K|| \beta(X). \tag{4}$$

(3) and (4) together give that

$$\beta(A(X)) = \beta(FrKG(X)) \le rb_1b_2||K||\beta(X). \tag{5}$$

For brevity, put $p = rb_1b_2||K||$ and recall that $p < 1$ by virtue of (h_4).

Now define a decreasing sequence (B_s^n) of nonempty, bounded, closed convex subsets of B_s that are invariant under A by putting $B_s^1 = \overline{\text{co}}\, A(B_s)$, $B_s^{n+1} = \overline{\text{co}}\, A(B_s^n)$ for $n \in \mathbf{N}$. Applying (5) it is easy to see that

$$\beta(B_s^{n+1}) \le p^{n+1}\beta(B_s) \quad n \in \mathbf{N}$$

and so

$$\lim_n \beta(B_s^n) = 0.$$

This implies (see [**4**]) that $Y = \cap_{n\in\mathbf{N}} B_s^n$ is a nonempty, closed, convex and relatively weakly compact (i.e., uniformly integrable) subset of B_s that is invariant under A also. Now, it is enough to show that $A(y)$ is relatively compact in order to conclude our proof with a simple application of the Schauder Fixed Point Theorem. Relative compactness of $A(y)$ can be proved exactly as in the last part of the main theorem of [**7**]. $\square$

In conclusion, we want to thank the referee for suggesting looking for more degrees of freedom by assuming, for instance, that $p, q \ge 1, G(L^1) \subseteq L^q, K(L^q) \subseteq L^p, F(L^p) \subseteq L^1$. We do not have the answer to this question in the above general situation; however, if we assume either

1) $|F(t,x)| \le h_1(t) + b\,|x|^r \quad t$ *a.e.* on $[0,1],\ x \in \mathbf{R},\ r < p$

or

2) $K(L^p) \subseteq L^q, q = 1$

we can repeat the proof of our theorem with $Y = Bs$.

Indeed, in both cases, the operator FK maps bounded subsets of L^p into uniformly integrable subsets of L^1. Hence the proof of the main Theorem in [**7**] can be used to show that $A(Y) = A(Bs)$ is relatively compact.

REFERENCES

1. J. Appell and E. De Pascale, *Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili*, Boll. Un Mat. Ital. Serie VI, **II-B**, (1984), 497–515.

2. J. Banas, *Integrable solutions of Hammerstein and Urysohn integral equations*, J. Austral. Math. Soc. **46** (1989), 61–68.

3. J. Banas and Z. Knap, *Integrable solutions of a functional-integral equation*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 31–38.

4. F.S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R.S. Roumanie **21** (1977), 259–262.

5. K. Deimling, *Nonlinear functional analysis*, Springer Verlag, New York, 1985.

6. G. Emmanuele, *About the existence of integrable solutions of a functional integral equation*, to appear in Rev. Mat. Univ. Complut. Madrid.

7. ———, *Integrable solutions of Hammerstein integral equations*, to appear in Appl. Anal.

8. M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Elmsford, 1964.

9. R.H. Martin, *Nonlinear operators and differential equations in Banach spaces*, Wiley and Sons, New York, 1976.

10. G. Scorza Dragoni, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova **17** (1948), 102–106.

11. P.P. Zabrejko, A.I. Koshelev, M.A. Krasnoselskii, S.G. Mikhlin, L.S. Rakovshchik and V.J. Stecenko, *Integral equations*, Noordhoff, Leyden, 1975.

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