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Gelfand-Phillips property in the completion of the space of Pettis integrable functions[1]

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Abstract. We consider the normed space $\mathcal{P}(\mu, X)$ of Pettis integrable functions with values in a Banach space X and we prove that if X has the Gelfand-Phillips property, then even the completion of $\mathcal{P}(\mu, X)$ has the same property.

Keywords: Pettis integrable functions, precompactness, Gelfand-Phillips property

Classification: 46E40, 46B20.

Introduction.

Let (S, Σ, μ) be a finite measure space and X a Banach space. We consider the normed space $\mathcal{P}(\mu, X)$ of all (μ)-Pettis integrable functions, with values in X, equipped with the norm

$$\|f\| = \sup\left\{ \int_S |x^* f(s)|\, d\mu : x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

We say that X has the Gelfand-Phillips property (see [1]) if any bounded subset M such that

(1) $$\lim_n \sup_M |x_n^*(x)| = 0 \ \textit{for any } w^*\textit{-null sequence } (x_n^*) \subset X^*$$

is relatively compact. A set verifying (1) will be called "limited".

Purpose of this note is to prove that if X has the Gelfand-Phillips property, then even the completion $\widehat{\mathcal{P}(\mu, X)}$ of $\mathcal{P}(\mu, X)$ has the same property.

In order to give our result we need the following remark done in [1].

Proposition 1. *If $f : S \to X$ is Pettis integrable and X has the Gelfand-Phillips property, then the set $\left\{\int_A f(s) d\mu : A \in \Sigma\right\}$ is relatively compact.*

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Proof: Using the $\mu-$continuity of the indefinite integral of f, together with the finiteness of μ, it is very easy to show that $\left\{\int_A f(s)d\mu : A \in \Sigma\right\}$ is limited. ▪

Result.

Our proof of the main result of the paper relies on the followinf theorem about the (strong) precompactness in the space $\mathcal{P}_c(\mu, X)$, the subspace of $\mathcal{P}(\mu, X)$ consisting of those f having an indefinite integral with compact range.

Theorem 1. *Let H be a bounded subset of $\mathcal{P}_c(\mu, X)$. If the following assumptions*
(i) the set $\{x^* f : x^* \in X^*, \|x^*\| \leq 1, f \in H\}$ is relatively compact in $L^1(\mu)$
(ii) the set $\{\int_S g(s)f(s)\, d\mu : g \in L^\infty(\mu), \|g\| \leq 1, f \in H\}$ is relatively compact in X
are verified, then H is precompact in $\mathcal{P}_c(\mu, X)$.

Proof: Choose $(f_n) \subset H$ and observe that under (i) and (ii), H is weakly precompact ([3]). Then we can assume, by passing to a subsequence if necessary, that (f_n) is weak Cauchy. Now, suppose that f_n has no Cauchy subseqeunces. There are $\eta > 0, (f_{n_h}), (f_{m_h})$ such that

$$\eta < \|f_{n_h} - f_{m_h}\| \qquad \text{for all } h \in N$$

For suitable sequences $(x^*_h) \subset X^*, \|x^*_h\| \leq 1, (g_h) \subset L^\infty(\mu), \|g_h\| \leq 1$, we have

$$\eta < \int_s g_h(s)(f_{n_h}(s) - f_{m_h}(s))x^*_h \, d\mu \qquad \text{for all } h \in N$$

Now, suppose that $(x^*_{h_\gamma})$ and (g_{h_γ}) are suitable subsnets weak* converging, respectively, to $x^* \in X^*, g \in L^\infty(\mu)$. Rewriting the last inequality for $(x^*_{h_\gamma})$ and (g_{h_γ}), we have

$$\eta < \int_s x^*_{h_\gamma}\, g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = \int_s x^*_{h_\gamma}\, g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu -$$
$$\int_s x^*\, g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu + \int_s x^*\, g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu -$$
$$\int_s x^*\, g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu + \int_s x^*\, g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu =$$
$$(x^*_{h_\gamma} - x^*) \int_s g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu +$$
$$\int_s x^*\, (f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, (g_{h_\gamma}(s) - g(s))\, d\mu +$$
$$\int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu$$

Now observe that the following limit relations are verified

(j) $\lim_{\gamma}(x^*_{h_\gamma} - x^*) \int_s g_{h_\gamma}(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = 0$, because $x^*_{h_\gamma} - x^* \xrightarrow{w^*} \theta$ and (ii) holds true

(jj) $\lim_{\gamma} \int_s x^* \left(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s)\right) \left(g_{h_\gamma}(s) - g(s)\right) d\mu = 0$, because $g_{h_\gamma} - g \xrightarrow{w^*} \theta$ and (i) holds true

(jjj) $\lim_{\gamma} \int_S x^* g(s)(f_{n_{h_\gamma}}(s) - f_{m_{h_\gamma}}(s))\, d\mu = 0$, because (f_n) is a weak Cauchy sequence. The reached contradiction gives our thesis.

Remark 1. It is possible to show that even the converse of Theorem 1 is true.

Remark 2. In a sense, the above result is the best possible; indeed, if H is a subset of $\mathcal{P}(\mu, X)$ (it doesn't matter how the range of the indefinite integral is) for which the above Theorem is true, then H must be a subset of $\mathcal{P}_c(\mu, X)$. This follows very easily from (ii) by choosing $g = \chi_A, A \in \Sigma$.

Now we are ready to give our main result

Theorem 2. *Assume that X has the Gelfand-Phillips property. Then $\mathcal{P}\widehat{(\mu, X)}$ has the same property.*

Proof: First of all, note that $\mathcal{P}\widehat{(\mu, X)} = \mathcal{P}_c\widehat{(\mu, X)}$, by virtue of Proposition 1. And so we have just to prove that $\mathcal{P}_c\widehat{(\mu, X)}$ enjoys the Gelfand-Phillips property. Let H be a limited subset of $\mathcal{P}_c\widehat{(\mu, X)}$ and (z_n) be a sequence in H. By virtue of the density of $\mathcal{P}_c(\mu, X)$ we can choose a sequence $(f_n) \subset \mathcal{P}_c\widehat{(\mu, X)}$ that is limited and such that $\lim_n \|z_n - f_n\| = 0$. It will be enough to show that (f_n) is relatively compact. This will be done by proving that (f_n) verifies (i) and (ii) of Theorem 1; then the completness of $\mathcal{P}_c\widehat{(\mu, X)}$ will do the remaining job. First of all, assume that the set $A = \{x^* f_n : x^* \in X^*, \|x^*\| \leq 1, n \in N\}$ is not limited in $L^1(\mu)$. There are $(g_h) \subset L^\infty(\mu), \|g_h\| \leq 1, g_h \xrightarrow{w^*} \theta, (x^*_h f_{n_h}) \subset A$ for which $\inf_h |g_h x^*_h f_{n_h}| > 0$.

Now, observe that $g_h x^*_h \in \left[\mathcal{P}_c\widehat{(\mu, X)}\right]^*$ for any $h \in N$ and furthermore $g_h x^*_h \xrightarrow{w^*} \theta$. This last assertion can be shown as it follows.

Take $f \in \mathcal{P}_c(\mu, X)$ and calculate $(g_h x^*_h)(f) = g_h(x^*_h f), h \in n$. Since $f \in \mathcal{P}_c(\mu, X)$, a result due to Edgar ([2]) tells us that $(x_h^* f)$ is relatively compact in $L^1(\mu)$ and so

$$\lim_h g_h(x_h^* f) = 0$$

because $g_h \xrightarrow{w^*} \theta$. Since $\mathcal{P}_c(\mu, X)$ is dense in $\mathcal{P}_c\widehat{(\mu, X)}$ we can conclude that $g_h x_h^* \xrightarrow{w^*} \theta$, as we wanted. Being (f_n) limited in $\mathcal{P}_c(\mu, X)$ (and so in $\mathcal{P}_c\widehat{(\mu, X)}$) we get a contradiction. Hence $\{x^* f_n : x^* \in X^*, \|x^*\| \leq 1, n \in N\}$ is limited in $L^1(\mu)$, a Banach space with the Gelfand-Phillips property. (i) of Theorem 1 is then true. Now we pass to (ii). Again, assume the set $\left\{\int_S g(s) f_n(s)\, d\mu : g \in L^\infty(\mu), \|g\| \leq 1, n \in N\right\}$ is not limited in X. There are a weak* null sequence $(x_h^*) \subset X^*, \|x_h^*\| \leq 1$ and $(g_h f_{n_h})$ such that $\inf_h |x_h^*(g_h f_{n_h})| > 0$. But once more $(g_h f_{n_h})$ is a weak* null sequence in $\left[\mathcal{P}_c\widehat{(\mu, X)}\right]^*$. Indeed, if $f \in \mathcal{P}_c(\mu, X)$ we have

$$\left|\int_S x_h^* g_h(s) f(s) d\mu\right| \leq \int_S |x_h^* g_h(s) f(s)| d\mu \leq \int_S |x_h^* f(s)| d\mu \qquad \text{for all} \qquad h \in N.$$

Now, observe that $x_h^* f \to 0$ almost uniformly. Putting $S_h^+ = \{s : x_h^* f(s) \geq 0\}$ and $S_h^- = \{s : x_h^* f(s) < 0\}, h \in N$, we get, for any $h \in N$,

$$(2) \quad \int_S |x_h^* f(s)| d\mu = \int_{S_h^+} x_h^* f(s) d\mu - \int_{S_h^-} x_h^* f(s) d\mu \leq \left|\int_{S_h^+} x_h^* f(s) d\mu\right| + \left|\int_{S_h^-} x_h^* f(s) d\mu\right|$$

Now, given $\epsilon > 0$ there is $A_\epsilon \in \Sigma, \mu(A_\epsilon^c) < \epsilon$, such that $x_h^* f \to 0$ uniformly on A_ϵ. On the other hand, the indefinite integral of f is $\mu-$continuous and so given $\gamma > 0$ there is $\delta > 0$ such that $\left\|\int_A f(s) d\mu\right\| < \gamma$ whenever $\mu(A) < \delta$. Take $\epsilon = \delta$. By (2) we have

$$\int_S |x_h^* f(s)| d\mu \leq \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s) d\mu\right| + \left|\int_{S_h^+ \setminus A_\delta} x_h^* f(s) d\mu\right| +$$

$$+ \left|\int_{S_h^- \cap A_\delta} x_h^* f(s) d\mu\right| + \left|\int_{S_h^- \setminus A_\delta} x_h^* f(s) d\mu\right| \leq$$

$$+ \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s) d\mu\right| + \left|\int_{S_h^- \cap A_\delta} x_h^* f(s) d\mu\right| +$$

$$+ \left\|\int_{S_h^+ \setminus A_\delta} f(s) d\mu\right\| + \left\|\int_{S_h^- \setminus A_\delta} f(s) d\mu\right\| +$$

$$+ \left|\int_{S_h^+ \cap A_\delta} x_h^* f(s) d\mu\right| + \left|\int_{S_h^- \cap A_\delta} x_h^* f(s) d\mu\right| + 2\gamma \leq 2 \int_{A_\delta} |x_h^* f(s)| d\mu + 2\gamma.$$

Since $x_h^* f \to 0$ uniformly on A_δ, we are done, i.e. we have reached the sought-for contradiction (use the density of $\mathcal{P}_c(\mu, X)$ in $\mathcal{P}_c\widehat{(\mu, X)}$, too). Being X a Banach space with the Gelfand-Phillips property, even (ii) in Theorem 1 is verified. The proof is complete.■

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