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`Abstract.`- We present some results showing that the Gelfand-Phillips property lifts from Banach spaces E and F to spaces of linear, bounded operators and to certain tensor products of Banach spaces.

1. Introduction

Let E be a Banach space. A (bounded) subset A of E is called *limited* if, for every weak* null sequence (x_n^*) in the dual space E*, we have $x_n^*(x) \to 0$ uniformly for x in A. If all limited subsets of E are relatively (norm) compact, then E is said to have the Gelfand-Phillips property (*see* [8]) or to be a Gelfand-Phillips space (sometimes we shall write E $\in$ (GP) for short). It is well known (*see* [25] for instance and its References) that Banach spaces with Schur property, separably complemented Banach spaces, Banach spaces with weak* sequentially compact dual balls, dual Banach spaces with the Weak Radon-Nikodym Property are Gelfand-Phillips spaces. Furthermore, l_∞ doesn't possess this property usually inherited by closed subspaces, so that no space containing l_∞ can have it. In recent years a lot of papers have been devoted to the study of this family of Banach spaces and, mainly, to the question of the construction of new Gelfand-Phillips spaces from old ones (*see* [8], [9], [14], [21], [25] and References therein). The present note, which can be seen as a continuation of [9], is concerned with this question, too. More precisely, we shall prove that if E, F are special spaces in (GP) then L(E,F) $\in$ (GP) (L(E,F) = Banach space of linear, bounded operators from E into F) as well as special tensor products of E and F do.

2. The Gelfand-Phillips property for spaces of operators

As far as we know, no results about the Gelfand-Phillips property in L(E,F) exist unless L(E,F) = K(E,F) (K(E,F) = Banach space of compact operators from E into F) that is a Gelfand-Phillips space if and only if E* and F are ([8]); this situation can be justified from the fact that in a lot of cases the hypothesis L(E,F) $\neq$ K(E,F) implies that c_0 embeds into K(E,F) and then l_∞ embeds into L(E,F).

Theorem 1.-*Let E, F be two Banach spaces satisfying one of the following conditions*
1) c_0 embeds into either F or E^*
2) F is a complemented subspace of a Banach space Z having an unconditional Schauder decomposition (Z_n) such that $L(E, Z_n) = K(E, Z_n)$ for all $n \in \mathbb{N}$
3) E is a $\mathscr{L}_\infty$ space and F a $\mathscr{L}_1$ space
4) $E = C[0, 1]$ and F has cotype 2
5) E is weakly compactly generated, F is a subspace of a space G with a shrinking unconditional basis and E^* or F^* has the bounded approximation property
7) E has an unconditional finite dimensional expansion of the identity.
If $L(E, F) \neq K(E, F)$ then $L(E, F)$ cannot have the Gelfand-Phillips property.

Proof. - Under the assumption L(E,F) $\neq$ K(E,F), results in [10], [12], [15], [17] imply that c_0 embeds into K(E,F) and l_∞ embeds into L(E,F). Since the Gelfand-Phillips property is inherited by closed subspaces and l_∞ doesn't have this property, we are done.

So, in particular, if E and F are "classical" Banach spaces, then L(E,F) has the Gelfand-Phillips property if and only if L(E,F) = K(E,F) and E* and F have it.

So, if one wants to get a new result on the Gelfand-Phillips property in L(E,F) (under the assumption L(E,F) $\neq$ K(E,F)), he has to assume that c_0 doesn't embed into K(E,F). As far as we know, there are just two examples of pairs E, F such that E*, F $\in$ (GP), L(E,F) $\neq$ K(E,F) and c_0 doesn't embed into K(E,F); one ([12]) is obtained by taking E = F = X, X a $\mathscr{L}_\infty$-space with Schur property constructed in [2] by Bourgain and Delbaen. For this space we have the following result

Theorem 2. - *Let $E^* \in (GP)$ and F have the Schur property. Then $L(E, F) \in (GP)$.*

Proof. - In order to prove that L(E,F) $\in$ (GP) it is enough to show that any w-null, limited sequence (A_n) $\subset$ L(E,F) is norm null ([9]). We assume by contradiction that there is a limited sequence (A_n) such that $A_n \xrightarrow{w} 0$ and $\| A_n \| = 1$ for all $n \in \mathbb{N}$. Let us consider $y^* \in F^*$ and the operator $A \to A^*(y^*)$ from L(E,F) into E^*; it is clear that $A_n^*(y^*) \xrightarrow{w} 0$ and $(A_n^*(y^*))$ is limited and so $\| A_n^*(y^*) \| \to 0$. So we have

$$|A_n(x_n)(y^*)| \leq \|A_n^*(y^*)\| \to 0$$

from which it follows that $A_n(x_n) \xrightarrow{w} 0$. Since F has Schur property, $\| A_n(x_n) \| \to 0$ a contradiction. We are done.

Corollary 3. - *Let E^* have the Schur property and $F^{**} \in (GP)$. Then $L(E,F) \in (GP)$.*

Proof. - The mapping $T \to T^*$ maps L(E,F) onto a closed subspace of L(F^*,E^*), that has the Gelfand-Phillips property by virtue of Theorem 2.

As we have already said the interest in Theorem 2 is due to the fact it is the first result we know about the Gelfand-Phillips property in L(E,F), under L(E,F) $\neq$ K(E,F). Actually, we can say more. There are a lot of results concerning isomorphic properties of Banach spaces in L(E,F) (*see* [1], [5], [6], [13], [19], [21] and References therein) when L(E,F) = K(E,F) but, as far as we know, this is the first time that an isomorphic property is shown to lift from E^* and F to L(E,F), when L(E,F) $\neq$ K(E,F).

We also remark that among those isomorphic properties that, up to now, are known to pass from E^* and F to K(E,F) under the assumption L(E,F) = K(E,F) there is the Radon-Nikodym Property ([1], [5], [22]). Sometimes that assumption is even necessary as results in [5] and [10] prove. However we know that at least in one case K(E,F) has the Radon-Nikodym Property even if L(E,F) $\neq$ K(E,F). Let us assume E = F = Y, where Y is the $\mathscr{L}_\infty$-space with the Radon-Nikodym Property such that Y^* is isomorphic to l_1 constructed in [2]. It is clear that L(E,F) $\neq$ K(E,F) = $E^* \otimes_\epsilon F$. But $E^* = Y^*$ is isomorphic to l_1 and so $E^* \otimes_\epsilon F$ is isomorphic to $l_1 \otimes_\epsilon F = K(c_0,F)$ a space that has the Radon-Nikodym Property if and only if F has the same property, as obtained by Diestel and Morrison in [5]. If we take E = F = Y, we get the second example of a pair such that E^*, F $\in$ (GP), L(E,F) $\neq$ K(E,F) and $c_0 \not\hookrightarrow$ K(E,F). We do not know if L(Y,Y) $\in$ (GP).

The next result is about the space of p-nuclear operators from E into F equipped with the p-nuclear norm (*see* [6]).

Theorem 4. - *Let E^* have the Radon-Nikodym Property and F be separably complemented. Then $N_p(E,F) \in (GP), 1 \leq p < \infty$.*

Proof. - Let (T_n) be a w-null, limited sequence in N_p(E,F). For each $n, m \in \mathbb{N}$ (and $T_0 = \theta$) choose $x^*_{n,m,j,k} \in E^*$, $y_{n,m,j,k} \in$F such that

$$\|T_n - T_m\| = \inf_j \left\{ \left(\sum_{k=1}^{\infty} \|x^*_{n,m,j,k}\|^p \right)^{1/p} \sup_{\|y^*\| \leq 1} \left(\sum_{k=1}^{\infty} \|y^*(y_{n,m,j,k})\|^{p'} \right)^{1/p'} \right\}$$

and

$$(T_n - T_m)(\bullet) = \sum_{k=1}^{\infty} x^*_{n,m,j,k}(\bullet)\, y_{n,m,j,k} \qquad \text{for all } j \in \mathbb{N}\ .$$

Consider F_0 separable and complemented in F such that F_0 contains each $y_{n,m,j,k}$; $N_p(E,F_0)$ is a closed, complemented subspace of N_p(E,F) containing (T_n) that is w-null and limited in it, too. So it is enough to show that $\|T_n\| \to 0$ in $N_p(E,F_0)$. Now, let us put W $= \overline{\text{span}}(x^*_{n,m,j,k})$ and take E_0 separable, closed subspace of E such that W $\subset E^*_0$. The restriction map **R** from $N_p(E,F_0)$ into $N_p(E_0,F_0)$ maps T_n into a w-null limited sequence of $N_p(E_0,F_0)$. Since E^* has the Radon-Nikodym property, E^*_0 is separable and so $N_p(E_0,F_0)$ is a separable space; hence $N_p(E_0,F_0) \in$ (GP) and $\|\mathbf{R}T_n\| \to 0$. The choice of E_0 also implies that **R** is an isometry on T_n, so that $\|T_n\| \to 0$ in $N_p(E,F_0)$. We are done.

2. The Gelfand-Phillips property for special tensor products

In this section we present some results concerning the Gelfand-Phillips property in certain tensor products of Banach spaces. Two of these results are concerned with the projective tensor product of Banach spaces (we refer to [6] for this well-known definition), whereas the other theorems use different tensor norms introduced in [23] and [18].

Theorem 5. - *Let E, F be two separably complemented Banach spaces. Then $E \otimes_\pi F$ has the Gelfand-Phillips property.*

Proof. - Let (x_n) be a limited subset of E$\otimes_\pi$F. For all $n \in \mathbb{N}$, there is $(z_{h,n})_{h\in\mathbb{N}}$ such that $\lim_h \|z_{h,n} - x_n\| = 0$ where $z_{h,n} = \sum_{i=1}^{p(h,n)} x_i^n \otimes y_i^n$. Obviously $(z_{h,n})_{h\in\mathbb{N}}$ is a limited subset of E$\otimes_\pi$F, too.

Since E and F are separably complemented we can choose two separable subspaces E_0 of E and F_0 of F such that $E_0 \supseteq \{x_i^n : i = 1, 2, ..., p(h, n); h, n \in \mathbb{N}\}$, $F_0 \supseteq \{y_i^n : i = 1, 2, ..., p(h, n); h, n \in \mathbb{N}\}$ and $P_1 : E \to E_0, P_2 : F \to F_0$ are projections. For each $z_{h,n}$ we have two norms: $\|z_{h,n}\|_0$ the norm of $z_{h,n}$ as element of $E_0\otimes_\pi F_0$ and $\|z_{h,n}\|_\pi$ the norm of $z_{h,n}$ as element of E$\otimes_\pi$ F; it is easy to show that there are $M_1, M_2 > 0$ such that

$$\|z_{h,n} - z_{k,m}\|_\pi \leq M_1\|z_{h,n} - z_{k,m}\|_0 \leq M_2\|z_{h,n} - z_{k,m}\|_\pi \qquad \text{for all } h, n, k, m \in \mathbb{N}$$

(use the fact that $(X \otimes_\pi Y)^* = L(X, Y^*)$, for arbitrary Banach spaces X and Y).

Furthermore, the operator $P_1 \otimes P_2$ maps $E \otimes_\pi F$ into $E_0 \otimes_\pi F_0$ in such a way that $(P_1 \otimes P_2)(z_{h,n}) = z_{h,n}$ for all $h, n \in \mathbb{N}$; this means that $(z_{h,n})_{h,n\in\mathbb{N}}$ is limited in $E_0 \otimes_\pi F_0$ a (separable and hence a) Gelfand-Phillips space. This means that $(z_{h,n})_{h,n\in\mathbb{N}}$ is limited in $E_0 \otimes_\pi F_0$ and so, thanks to the above inequalities about $\| \bullet \|_0$, even in $E \otimes_\pi F$. This implies that (x_n) is relatively compact in $E \otimes_\pi F$. We are done.

In the paper [23] P. Saphar introduced certain tensor norms $d_p, 1 < p < \infty$, such that $(E \otimes_{d_p} F)^* =$\{ absolutely p'-summing operators T:E $\to$ F* equipped with the absolutely summing norm\}, $1/p + 1/p' = 1$. Using the same proof of Theorem 5 we are able to show that following result

Theorem 6. - *Let E, F be separably complemented Banach spaces. Then $E \otimes_{d_p} F \in$ (GP).*

Theorem 7. - *Let E, F contain no copy of l_1. Let us assume E^* has both the Radon-Nikodym Property and the metric approximation property. Then $E^* \otimes_\pi F^* \in$ (GP).*

Proof. - Under our assumptions $E \otimes_\pi F$ does not contain l_1 (*see* [22]) and so $(E \otimes_\epsilon F)^*$ has the Weak Radon-Nikodym property and it is a Gelfand-Phillips space (*see* [25]). Furthermore, $E^* \otimes_\pi F^*$ is a closed subspace of $(E \otimes_\epsilon F)^*$ ([16]) and so it inherites the Gelfand-Phillips property. We are done.

Remark 1. - Theorem 7 actually shows that $E^* \otimes_\epsilon F^*$ has the so called (DPrcP), considered in [13].

Now, let us consider a different tensor norm introduced by Levin in [18]. If E is a Banach lattice and F a Banach space, for $z = \sum_{i=1}^{p} x_i \otimes y_i$, Levin put

$$n_E(z) = \inf\left\{ \|u\|_E : u \geq \left| \sum_{i=1}^{p} x_i \ < y_i, y^* > \right| : y^* \in F^*, \|y^*\| \leq 1 \right\}.$$

He denoted by E $\widetilde{\otimes}$ F the completion of (E $\otimes$ F, n_E).

We have the following result

Theorem 8. - *Let E be σ-complete. Then E $\widetilde{\otimes}$ F ∈ (GP), provided E, F ∈ (GP).*

Proof. - Since E ∈ (GP), E cannot contain l_∞ and so it is an order continuous Banach lattice ([20]). Let (x_n) be a limited sequence in E $\widetilde{\otimes}$ F; for all $n \in \mathbb{N}$, choose a sequence $(z_{hn})_{n\in\mathbb{N}}$ in E $\otimes$ F such that $n_E(x_n - z_{hn}) \to 0$ as $h \to \infty$. But $z_{hn} = \sum_{i=1}^{p(h,n)} e_i^n \otimes y_i^n$, for all $h, n \in \mathbb{N}$. Let E$_0$ be a band containing the subspace spanned by $\{e_i^n : i = 1, 2, ..., p(h, n), h, n \in \mathbb{N}\ \}$; it is known ([20]) that E$_0$ can be chosen complemented in E; furthermore, E$_0$ is an order continuous Banach lattice with a weak unit and the projection P:E $\xrightarrow{onto}$ E$_0$ is a positive operator. It is clear that $(x_n) \subset$ E$_0\widetilde{\otimes}$ F, a closed subspace of E $\widetilde{\otimes}$ F ([18]). Now we show it is possible to define a projection Q: E $\widetilde{\otimes}$ F $\xrightarrow{onto}$ E$_0$ $\widetilde{\otimes}$ F. For $z = \sum_{i=1}^{p} x_i \otimes y_i \in$ E $\otimes$ F we put $Q(z) = \sum_{i=1}^{p} P(x_i) \otimes y_i$. It will be enough to show that Q is continuous, because it is clear that Q is linear and Q restricted to E$_0\otimes$ F is the identity. From Lemma 5 in [18] it follows that, under our hypothesis,

$$\sup_{\|y^*\|\leq 1} \left| \sum_{i=1}^{p} x_i < y_i, y^* > \right|$$

exists in E and that

$$n_E(z) = \left\| \sup_{\|y^*\|\leq 1} \left| \sum_{i=1}^{p} x_i < y_i, y^* > \right| \right\|_E .$$

We have, for $y^* \in F^*, \|y^*\| \leq 1$,

$$\left| \sum_{i=1}^{p} P(x_i) < y_i, y^* > \right| \leq |P| \left| \sum_{i=1}^{p} x_i < y_i, y^* > \right| \leq |P| \sup_{\|y^*\|\leq 1} \left| \sum_{i=1}^{p} x_i < y_i, y^* > \right|$$

and so

$$\sup_{\|y^*\|\leq 1} \left| \sum_{i=1}^{p} P(x_i) < y_i, y^* > \right| \leq |P| \sup_{\|y^*\|\leq 1} \left| \sum_{i=1}^{p} x_i < y_i, y^* > \right| .$$

Since $|P|$ is a linear, bounded operator ([24]) we get easily

$$n_E(Q(z)) \leq \| \, |P| \, \| \, n_E(z) \qquad\qquad z \in E \otimes F.$$

Now, we can extend Q to all of $E_0 \widetilde{\otimes}$ F by continuity, so obtaining the required projection. We also recall that E_0 is isometrically isomorphic to a Köthe space E_1 on a probability space and that $E_0 \widetilde{\otimes} F$ is isometrically isomorphic to the Köthe space $E_1(F)$ of vector valued functions ([3]). Now, it is enough to apply a recent result in [14] showing that $E_1(F) \in$ (GP). We are done.

The author wishes to thank Prof. Hensgen for pointing out a mistake in a first proof of Theorem 8.

Remark 2. - The same procedure of "locally embedding" of E $\widetilde{\otimes}$ F into a Köthe space of vector valued functions can be used to show the following facts

(i) E $\widetilde{\otimes}$ F contains l_∞ iff either E or F does (use [11])

(ii) E $\widetilde{\otimes}$ F contains c_0 iff either E or F does (use [7])

(iii) E $\widetilde{\otimes}$ F contains l_1 iff either E or F does (use [16])

(iv) E $\widetilde{\otimes}$ F has The Radon-Nikodym Property iff E and F have it (use [4]).

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