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Existence of Approximate Solutions for O. D. E.'s under Caratheodory Assumptions in Closed, Convex Sets of Banach Spaces

By

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1. Introduction

Let $I=[0, 1]$ and X be a closed, convex subset of a Banach space E. Assume that $f: I\times X\rightarrow E$ is a function verifying the following Caratheodory assumptions
(C1) the functions $t\rightarrow f(t, x)$ are strongly measurable, for any $x\in X$,
(C2) the functions $x\rightarrow f(t, x)$ are continuous, for almost all $t\in I$,
and consider the following Cauchy problem

$$\text{(CP)}\qquad \begin{cases} x'=f(t, x)\\ x(0)=x_0\end{cases}$$

where $x_0\in X$.

In order to get a solution of (CP) essentially two methods have been used, so far: the first one uses fixed point theorems to be applied to suitable operators connected with f and the second one uses the existence of suitable (and with good properties) approximate solutions, i.e., sequences of (usually equicontinuous absolutely continuous a.e. derivable) functions $x_n: J\rightarrow E$, J a suitable subinterval of I, such that

$$\lim_n \|x_n'(t)-f(t, x_n(t))\|=0 \quad \text{a. e. on } J;$$

In such a case one has only to consider conditions forcing a subsequence of $\{x_n\}$ to converge.

In this paper we are interested just in constructing a sequence of approximate solutions. There are several kinds of such a sequence; for instance, if $\overset{\circ}{X}\neq\emptyset$, Peano-Tonelli approximations work quite well to have a solution of (CP) under reasonable hypotheses on f (see [4], p. 113; [15], [21]); but if $\overset{\circ}{X}=\emptyset$, they are useless (because the construction cannot be performed in

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such a case) and then the Euler-Cauchy approximations (see [4] or [12]) have been used by several authors very successfully; but their construction requires the assumption "f is continuous".

Our main advance here is the following: the continuity hypothesis about f can be relaxed in special cases (for instance, when X is separable) assuming only that f satisfies (C1) and (C2), in closed, convex subsets X of E (even if $\overset{\circ}{X} = \emptyset$); we underline that Caratheodory assumptions are more natural than continuity hypotheses for (CP) and that in many occurrences of applications of (CP) to other problems (like P. D. E.'s or integro-differential equations, see [4], [17], [22]) the involved function f is not defined in balls, but just in subsets with empty interior. However, we are able to construct Euler-Cauchy approximations for (CP), at least in some special case; as far as we know, this is the first attempt in this direction.

We have also to say that Schechter ([18]) constructed different kinds of approximate solutions under (C1) and (C2), quite recently; but he also needs an assumption like (9) below (more restrictive than (9)) whereas we don't use it in our construction.

As a consequence of our main result we present two (partial) improvements of results due to von Harten-Mönch ([9]) and Song ([19]), about the existence of a solution to (CP) and to Deimling ([4], p. 114) about the existence of a unique solution; in this last case we are able to eliminate an assumption of uniform continuity about f that is quite restrictive in infinite dimensional Banach spaces. At the end of the note the cited result from [4] will be further generalized (when $\overset{\circ}{X} \neq \emptyset$ as in [4]), but in a more strict class of Banach spaces, always dispensing with the uniform continuity of f. These last two results will be easy consequences of theorems concerning perturbed Cauchy problems, improving theorems by Hu Shou Chuan ([10]) and Martin ([12]).

2. The main result

This section is devoted to present the result about existence of approximate solutions for (CP). We need the well known definitions of locally finite partition of unity and Dugundji system, for which we refer to [1]. We shall also use the following result of existence of approximate solutions in the case f continuous, that can be found in [4] and [12]; we enunciate it as a lemma.

Lemma 1. *Let I, X, E, x_0, f be as in the Introduction. Consider $r > 0$, $B = B(x_0, r) = \{x \in E, \|x - x_0\| \leqq r\}$, $X_r = X \cap B$. Assume that $f \colon I \times X_r \to E$ is continuous and bounded by, say, b. Moreover, we suppose that*

(1) $$\lim_{h \to 0^+} h^{-1} d(x + hf(t, x), X_r) = 0 \quad \textit{for} \quad t \in I,\ x \in X_r.$$

Let now $\delta < \min\{1, r/b\}$ *be. Take* $\varepsilon_0 > 0$ *such that* $(b + \varepsilon_0)\delta \leqq r$ *and consider a sequence of positive numbers* $\{\varepsilon_n\}$ *with* $\varepsilon_n < \varepsilon_0$ *and* $\varepsilon_n \to 0$. *Then, for each* $n \in N$, *there is an absolutely continuous function* $x_n \colon J = [0, \delta] \to X$ *having a derivative almost everywhere and such that*

$$\|x_n(t') - x_n(t'')\| \leqq (b + \varepsilon_0)|t' - t''| \quad \text{for} \quad n \in N,\ t',\ t'' \in J \tag{2}$$

$$\|x_n'(t) - f(t, x_n(t))\| \leqq \varepsilon_n \quad \text{for a.e. } t \in J. \tag{3}$$

Now we are ready to prove our first result.

Theorem 1. *Let* I, X, E, x_0, f *be as in the Introduction. Assume that*

a) *the functions* $t \to f(t, x)$ *are strongly measurable, for all* $x \in X$

b) *the functions* $x \to f(t, x)$ *are continuous, for almost all* $t \in I$

c) f *is bounded by, say,* b.

If r *and* X_r *are like in Lemma* 1, *assume* (1) *is verified, too. Taking* ε_0 *and* J *as in Lemma* 1, *the conclusion of that Lemma is true if we change* (3) *with the following*

$$\lim_n \|x_n'(t) - f(t, x_n(t))\| = 0 \quad \text{a.e. in } J \tag{3'}$$

provided f *verifies the Scorza-Dragoni assumption*

(SD) *for each* $\eta > 0$ *there is a closed subset* I_η *of* I *such that* $m(I \backslash I_\eta) < \eta$ *and* $f|_{I_\eta \times X}$ *is continuous.*

Proof. By virtue of (SD), given $k \in N$ there is a closed subset I_k of I such that $m(I \backslash I_k) < 1/k$ and $f|_{I_k \times X}$ is continuous. Of course I and I_k verify the well-known assumptions implying the existence of a Dugundji system $\{U_s, t_s\}_{s \in S}$ for $I \backslash I_k$ and of a locally finite partition of the unity $\{b_s\}_{s \in S}$ inscribed into $\{U_s\}_{s \in S}$ (see [1]). We define a new function $f_k \colon I \times X \to E$ by putting

$$f_k(t, x) = \begin{cases} f(t, x) & t \in I_k,\ x \in X \\ \sum b_s(t) f(t_s, x) & t \notin I_k,\ x \in X. \end{cases}$$

Of course f_k is an extension of $f|_{I_k \times X}$ and it is bounded by, again, b and continuous. We shall show that

$$\lim_{h \to 0^+} h^{-1} d(x + h f_k(t, x), X_r) = 0 \quad \text{for} \quad t \in I,\ x \in X_r. \tag{4}$$

Indeed, if $(t, x) \in I_k \times X$ (4) follows from the definition of f_k and (1). Let $t \notin I_k$, $x \in X$ be. For a finite number of $s \in S$ we have

$$f_k(t, x) = \sum b_s(t) f(t_s, x).$$

Let $\sigma > 0$ be. There is $\rho > 0$ such that if $0 < h < \rho$ we have

$$h^{-1} d(x + hf(t_s, x), X_r) < \sigma.$$

So we obtain a finite number of elements $y_s \in X_r$ such that

$$h^{-1} \| (x + hf(t_s, x)) - y_s \| < \sigma \quad \text{with } 0 < h < \rho. \tag{5}$$

By using the convexity of X_r we obtain $z = \sum b_s(t) y_s \in X_r$. Then we have, by virtue of (5),

$$\begin{aligned} & h^{-1} \| (x + hf(t_s, x)) - z \| \\ & \qquad \leqq \sum b_s(t)(h^{-1} \| (x + hf(t_s, x)) - y_s \|) < \sigma \text{ with } 0 < h < \rho. \end{aligned}$$

So (4) is true. We can apply Lemma 1 to f_k, for each $k \in \boldsymbol{N}$, and we obtain an absolutely continuous function $x_k \colon J \to X$, derivable almost everywhere, verifying (2) and (3) (in (3) read f_k instead of f). For $k \in \boldsymbol{N}$, we have

$$\begin{aligned} & \int_J \| x_k'(t) - f(t, x_k(t)) \| \, dt \\ & \leqq \int_J \| x_k'(t) - f_k(t, x_k(t)) \| \, dt + \int_J \| f_k(t, x_k(t)) - f(t, x_k(t)) \| \, dt \\ & \leqq m(J)\varepsilon_k + \int_{J \setminus I_k} \| f_k(t, x_k(t)) - f(t, x_k(t)) \| \, dt \\ & \leqq m(J)\varepsilon_k + m(J \setminus I_k) 2b. \end{aligned}$$

As $k \to \infty$, the last member tends to zero. Hence we obtain

$$\lim_k \| x_k'(\cdot) - f(\cdot, x_k(\cdot)) \|_{L^1} = 0. \tag{6}$$

It is well known that (6) implies the existence of a subsequence of $\{x_k\}$ verifying our thesis. The proof is over. ■

In Theorem 1 we have used the assumption (SD). Now we present two cases in which it is verified.

(SD1) *Let I, E be as in the Introduction. Assume X is a separable metric space and $f \colon I \times X \to E$ verifies* (a) *and* (b) *of Theorem* 1. *Then* (SD) *is verified.*

The proof of this can be found in [16].

(SD2) *Let I, X, E be as in the Introduction. Assume X is closed and convex, $f \colon I \times X \to E$ verifies* (a) *and* (b) *of Theorem* 1. *Suppose again that there are two functions L from I into E and H from X into $\boldsymbol{R}$ such that*

(7) $\quad L \in L^1(I, E)$ *and H is bounded on bounded sets*

(8) $\|f(t', x) - f(t'', x)\| \leqq \|L(t') - L(t'')\| |H(x)|(1 + \|f(t', x)\|) \qquad t', t'' \in I, \ x \in X.$

Then (SD) *is verified.*

We omit the simple proof which relies on an application of Lusin Theorem ([6]) to L. The assumption (8) was successfully already considered by a number of authors mainly in the study of nonlinear evolution equations in Banach spaces (see [3], [8], [11], [14]).

3. Some applications to existence results for (CP)

In section 2 we defined approximate solutions of Euler-Cauchy type for (CP) under Caratheodory assumptions in closed, convex sets X (with $\overset{\circ}{X} = \emptyset$, too). The availability of these approximations allows us to improve two existence results for (CP) due to von Harten-Möñch ([9]) (see, also Song [19]) and Deimling ([4]); those authors proved their theorems using the existing (at that time) sequences of approximations: for this reason in [9] it was assumed "f continuous" and in [19] "f Caratheodory, but $\overset{\circ}{X} = \emptyset$"; further, in [4] Deimling was forced to assume that f is uniformly continuous with respect to the second variable, by the nature of the Peano-Tonelli approximations he used. Our main gain here seems to be the following: we can assume "f Caratheodory with X closed and convex„ but even when $\overset{\circ}{X} = \emptyset$", so dispensing with the nonemptiness of $\overset{\circ}{X}$ and with uniform continuity; but, we recall this is possible not in as general as possible case: we have to assume that X is separable.

First of all we apply Theorem 1 to the results due to von Harten and Mönch ([9]) and Song ([19]). We observe that the proof of von Harten-Mönch result works with any kind of approximants as Peano-Tonelli approximants or Euler-Cauchy approximants (and for this reason we shall only state our result without proof); so whereas they were forced to assume f continuous, in order to use the Euler-Cauchy approximants constructed by Martin (see [4] or [12]) just when f is continuous, Song ([19]) improved their results assuming Caratheodory assumptions but with $\overset{\circ}{X} \neq \emptyset$, in such a way that Peano-Tonelli approximants could be used. Now, thanks to our main construction we can restate the above quoted theorems as it follows (α will denote the Kuratowski measure of non compactness and β the Hausdorff measure of non compactness, [4])

Theorem 2. *Let I, X, E, x_0, f be as in Theorem* 1. *Further, assume that for each bounded subset Y of X we have $\beta(f(t, Y)) \leqq \omega(t, \beta(Y))$ [resp. $\alpha(f(t, Y)) \leqq \omega(t, \alpha(Y))$] if E is weakly compactly generated space [resp. if E is a general Banach space] and ω from $I \times \boldsymbol{R}^+$ into $\boldsymbol{R}^+$ verifies Caratheodory assumptions and is such that the only non negative absolutely continuous function $u\colon I \to \boldsymbol{R}^+$*

for which $u(0) = 0$, $u'(t) \leqq \omega(t, u(t))$ [*resp.* $u'(t) \leqq 2\omega(t, u(t))$] *is* $u(t) = 0$ *on* I.
Then (CP) *has a solution on a suitable subinterval* J *of* I.

Observe that if E is a separable Banach space our result is strictly more general than those ones in [9] and [19].

The second application is a (partial) improvement of Theorem 8.1 of [4]. That theorem, obtained in general Banach spaces with $\mathring{X} \neq \emptyset$, uses Peano-Tonelli approximants under Caratheodory hypotheses; we already said that this forces the author to assume that f is uniformly continuous with respect to x, quite a restrictive requirement in infinite dimensional Banach spaces. Using our construction we are now able to dispense and with the nonemptiness of $\mathring{X}$ and with uniform continuity (not in general Banach spaces, but at least in separable ones). If $\{x_n\}$ is like in Theorem 1, we denote by Z the (bounded) subset $\{x_n(t): t \in I,\ n \in \boldsymbol{N}\}$ of X.

Theorem 3. *Let* I, X, E, x_0, f *be as in Theorem* 1. *Assume that* f *satisfies the following assumption of dissipative type*

(9) $\langle f(t, x) - f(t, y), x - y\rangle_- \leqq \omega(t, \|x - y\|)\|x - y\|$ *for* $t \in I$, $x, y \in Z$

where ω *is a uniqueness function as in Theorem* 2 (*we refer to* [4] *for the definition of* $\langle \cdot, \cdot \rangle_-$). *Then* (CP) *has a unique solution on a suitable subinterval* J *of* I.

Theorem 3 is an easy consequence of the following more general result about perturbed Cauchy problems (take $B = 0$ in Theorem 4).

Theorem 4. *Let* I, X, E, x_0, f *be as in Theorem* 1. *Assume that* $f = A + B$ *where* $A: I \times X \to E$, $B: I \times X \to E$ *verify* (C1), (C2) *and the following other hypothesis*

(10) *there exist two functions* φ_A, $\varphi_B \in L^1(I, \boldsymbol{R})$ *such that* $\|A(t, x)\| \leqq \varphi_A(t)$, $\|B(t, x)\| \leqq \varphi_B(t)$ *for a. a.* $t \in I$ *and all* $x \in X$.
Assume that A *verifies an assumption like* (9) *of Theorem* 3 *and that*

(11) $B(t, Z)$ *is relatively compact, for almost all* $t \in I$

(12) *for each* $\varepsilon > 0$ *there is a* (*closed*) *subset* I_ε *of* I, $m(I \backslash I_\varepsilon) < \varepsilon$ *such that* $B|_{I_\varepsilon \times Z}$ *is uniformly continuous.*
Then the perturbed Cauchy problem

(PCP) $$\begin{cases} x' = A(t, x) + B(t, x) \\ x(0) = x_0 \end{cases}$$

has a solution on J.

Proof. Put $f_k(\cdot) = B(\cdot, x_k(\cdot))$, $k \in \boldsymbol{N}$. First of all we shall prove that for all $p \in \boldsymbol{N}$ there is a sequence $\{f_k^p\}$ of $\{f_k\}$ such that

$$f_k^{p+1} \text{ is a subsequence of } \{f_k^p\} \tag{13}$$

$$\int_J \|f_n^p(t) - f_m^p(t)\| \, dt < 1/p \quad \text{for all} \quad n,\ m \in \boldsymbol{N}. \tag{14}$$

Assume $\{f_k^p\}$ has already been found. We shall show how to get $\{f_k^{p+1}\}$. We recall that $\|f_k(t)\| \leqq \varphi_B(t)$ for all $k \in \boldsymbol{N}$, t a.e. in J. Fix $1/(3(p+1))$ and consider $\varepsilon > 0$ such that $\int_A \varphi_B(t)\, dt < (1/3(p+1))$ if $m(A) < \varepsilon$. By (12) there is a closed I_ε, $m(I \backslash I_\varepsilon) < \varepsilon$ such that $B|_{I_\varepsilon \times Z}$ is uniformly continuous. Hence $\{f_k^p|_{I_\varepsilon}\}$ is a sequence of equicontinuous functions for which $\{f_k^p(t)\}$ is relatively compact in E. The Ascoli-Arzela Theorem implies that there is a subsequence $\{f_k^{p+1}\}$ of $\{f_k^p\}$ which is a Cauchy sequence in $C(I_\varepsilon, E)$. So for n, m sufficiently large we may assume that $\|f_n^{p+1}|_{I_\varepsilon} - f_m^{p+1}|_{I_\varepsilon}\|_{C(I_\varepsilon, E)} < 1/(3(p+1))$. From the above reasoning we have

$$\begin{aligned}
&\int_J \|f_n^{p+1}(t) - f_m^{p+1}(t)\| \, dt \\
&= \int_{J \backslash I_\varepsilon} \|f_n^{p+1}(t) - f_m^{p+1}(t)\| \, dt + \int_{J \cap I_\varepsilon} \|f_n^{p+1}(t) - f_m^{p+1}(t)\| \, dt \\
&\leqq 2 \int_{J \backslash I_\varepsilon} \varphi_B(t)\, dt + m(J \cap I_\varepsilon) \|f_n^{p+1}|_{I_\varepsilon} - f_m^{p+1}|_{I_\varepsilon}\|_{C(I_\varepsilon, E)} \\
&< 1/(p+1).
\end{aligned}$$

Of course $\{f_k^k\}$ is a subsequence of $\{f_k\}$ which is a Cauchy sequence in $L^1(J, E)$. Without loss of generality we can suppose that $\{f_k\}$ is Cauchy in $L^1(J, E)$. Put now $p_{nm}(t) = \|x_n(t) - x_m(t)\|$, $n, m \in \boldsymbol{N}$, $t \in J$. By well known properties of $\langle \cdot, \cdot \rangle_-$ we have, for n, $m \in \boldsymbol{N}$, $t \in J \backslash J_0$, $m(J_0) = 0$ where $J_0 = J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5$, being $J_1 = \{t \in J\colon \text{(C1) is true on } J/J_1\}$, $J_2 = \{t \in J\colon \text{(C2) is true on } J \backslash J_2\}$, $J_3 = \{t \in J\colon \|A(t, x)\| \leqq \varphi_A(t),\ \|B(t, x)\| \leqq \varphi_B(t) \text{ are true on } J \backslash J_3\}$, $J_4 = \{t \in J\colon \text{(9) and (11) are true on } J \backslash J_4\}$, $J_5 = \{t \in J\colon p'_{nm} \text{ exists for each } n, m \in \boldsymbol{N}\}$

$$\begin{aligned}
p'_{nm}(t) p_{mn} &\leqq \langle x'_n(t) - x'_m(t),\ x_n(t) - x_m(t) \rangle_- \\
&\leqq \langle A(t, x_n(t)) - A(t, x_m(t)),\ x_n(t) - x_m(t) \rangle_- \\
&\quad + p_{nm}(t) \{ \|B(t, x_n(t)) - B(t, x_m(t))\| + \|h_n(t)\| + \|h_m(t)\| \}
\end{aligned}$$

where $h_k(t) = x'_k(t) - [A(t, x_k(t)) + B(t, x_k(t))]$, $k \in \boldsymbol{N}$, t a.e. in J. Of course

$\|h_k\|_{L^1} \to 0$. If $T > \max_{n,m,t}|p_{nm}(t)|$ we have

$$p'_{nm}(t)p_{nm}(t) \leqq \omega(t, p_{nm}(t))p_{nm}(t) + T(\|f_n(t) - f_m(t)\| + \|h_n(t)\| + \|h_m(t)\|),$$
$$n, m \in N, \ t \in J \setminus J_0.$$

From now on we can proceed as in our paper [7] to obtain that at least a subsequence of $\{x_k\}$ must converge in $C(J, E)$. Of course the limit will be a solution for (CP). We are done. ■

Remark 1. Assumption (12) could seem too restrictive. However it allows quite a bad behaviour of B, in special cases. For instance, assume E is a reflexive and separable Banach space and X is a convex subset of E (or, more generally, X is weakly closed). Also suppose B verifies (11), (C1) and the following

(C2)′ *the functions* $x \to B(t, x)$ *are weakly-weakly sequentially continuous for almost all* $t \in I$,

a hypothesis used by several authors before ([2], [5], [13], [20]).

It is very easy to prove that (11) and (C2)′ alike imply that

(C2)″ *the functions* $x \to B(t, x)$ *are weakly-strongly sequentially continuous on* $\bar{Z}^{\omega}$ *(the weak closure of* Z*), for almost all* $t \in I$.

Now, we note that Z is a bounded subset and hence $\bar{Z}^{\omega}$ with the weak topology actually is a compact metric space (we denote by d the induced metric). If we consider B as a function defined in $I \times (\bar{Z}^{\omega}, d)$ with values into $(E, \|\cdot\|)$, we can apply the main result of [16] to prove that for each $\varepsilon > 0$ there is a (closed) subset I_ε of I, $m(I \setminus I_\varepsilon) < \varepsilon$, such that B restricted to $I_\varepsilon \times (\bar{Z}^{\omega}, d)$ is continuous; since $I_\varepsilon \times (\bar{Z}^{\omega}, d)$ is a compact space, then B restricted to it has to be uniformly continuous; this easily implies (12).

Observe that we could only assume E reflexive, without separability assumption, because of the following remark: Z is contained in a separable closed subspace Y of E, hence we could apply our ideas to this last space Y that is reflexive and separable.

Remark 2. The proof of Theorem 1 shows that the assumptions (9), (11), (12) are to be used just in order to look for a subsequence of $\{x_n\}$ converging; actually we don't need them in the construction of $\{x_n\}$. For this reason we assumed them only involving Z and not all of X; this fact is useful, sometime, as we see from Remark 1, where we didn't suppose X bounded in order to have (11), because it involves only Z that is, already, bounded by construction. We also observe that (12) implies that, for almost all $t \in I$, the functions $x \to B(t, x)$ have to be uniformly continuous just on Z, not on the all of X. Observe that Theorem 4 improves results by Hu Shou Chuan ([10]) and Martin ([12]) about the same kind of (PCP).

4. Another result about (PCP) (not using Theorem 1)

In this section, we want to state another result concerning (PCP) in the case of $\overset{\circ}{X} \neq \emptyset$; in this case, we can use a different kind of approximate solutions, the so-called Peano-Tonelli approximants defined by

$$x_n(t) = \begin{cases} x_0 & 0 \leqq t \leqq \dfrac{a}{n} \\ x_0 + \displaystyle\int_0^t f\left(s, x_n\left(s - \frac{a}{n}\right)\right) ds. & \dfrac{a}{n} \leqq t \leqq a \end{cases} \tag{15}$$

when $f = A + B$ verifies Caratheodory hypotheses (see [4]).

Theorem 5. *Let I, $\overset{\circ}{X}$, E, A, B be as in Theorem 4. Assume E^* is uniformly convex. Consider $x_0 \in \overset{\circ}{X}$ and a ball $B(x_0, r) \subset X$. Suppose there are two functions φ_A, $\varphi_B \in L^1(I, \boldsymbol{R})$ such that $\|A(t, x)\| \leqq \varphi_A(t)$, $\|B(t, x)\| \leqq \varphi_B(t)$, $t \in I$, $x \in X$. Let $J = [0, a]$ be, where $a < 1$ and $\int_0^a \varphi(s) ds \leqq r$, $\varphi = \varphi_A + \varphi_B$. Define a sequence $\{x_n\}$ like in* (15) *and assume that A, B verify* (C1) *and* (C2). *If* (9), (11), (12) *of Theorem* 4 *are true, then* (PCP) *has a solution on J.*

Proof. We can proceed as in Theorem 4 to show that the sequence $\{B(\cdot, x_n(\cdot))\}$ is relatively compact in $L^1(J, E)$, so we can assume (by passing to a subsequence if necessary) it is a Cauchy sequence.

Hence, with the same notation of Theorem 4 we have

$$p_{nm}(t)p'_{nm}(t) \leqq \left\langle A\left(t, x_n\left(t - \frac{a}{n}\right)\right) - A\left(t, x_m\left(t - \frac{a}{m}\right)\right), x_n(t) - x_m(t) \right\rangle_- \\ + p_{nm}(t)\{\|B(t, x_n(t)) - B(t, x_m(t))\| + \|h_n(t)\| + \|h_m(t)\|\}.$$

Since $\langle \cdot, \cdot \rangle_-$ is uniformly continuous on bounded subsets of $E \times E$ (see [4]) for $\varepsilon > 0$ and n, m sufficiently large, we have

$$p_{nm}(t)p'_{nm}(t)$$

$$\leqq \left\langle A\left(t, x_n\left(t - \frac{a}{n}\right)\right) - A\left(t, x_m\left(t - \frac{a}{m}\right)\right), x_n\left(t - \frac{a}{n}\right) - x_m\left(t - \frac{a}{m}\right) \right\rangle_- \\ + \varepsilon + p_{nm}(t)\{\|B(t, x_n(t)) - B(t, x_m(t))\| + \|h_n(t)\| + \|h_m(t)\|\}$$

$$\leqq \omega_A\left(t, \left\|x_n\left(t - \frac{a}{n}\right) - x_m\left(t - \frac{a}{m}\right)\right\|\right)\left\|x_n\left(t - \frac{a}{n}\right) - x_m\left(t - \frac{a}{m}\right)\right\| \\ + \varepsilon + p_{nm}(t)\{\|B(t, x_n(t)) - B(t, x_m))\| + \|h_n(t)\| + \|h_m(t)\|\};$$

again using the uniform continuity of $u \to \omega_A(t, u)u$ on bounded sets we get, for n, m sufficiently large,

$$p_{nm}(t)p'_{nm}(t) \leqq \omega_A(t, p_{nm}(t))p_{nm}(t) + 2\varepsilon + p_{nm}(t)\{\|B(t, x_n(t)) - B(t, x_m(t))\| + \|h_n(t)\| + \|h_m(t)\|\}.$$

Now it easy to see that $\{x_n\}$ admits a converging subsequence, using the arbitrariness of ε and the same proof of the result in [7]. So our proof is complete. ■

In the case of $B = 0$ we generalize partially a previous result by Deimling ([4], p. 114) by dispensing with the uniform continuity assumed in [4].

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