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Convergence of Successive Approximations for Implicit Ordinary Differential Equations in Banach Spaces

By

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§ 1. Introduction.

Let E be a Banach space and let x^0, y^0 be two points of E. If $B_1=\{x, x \in E, \|x-x^0\| \leqslant b_1\}$, $B_2=\{y, y \in E, \|y-y^0\| \leqslant b_2\}$, $b_1, b_2 \in R^+$, $I=[0, a] \subseteq R$, $a \in R^+$, and $F, F: I \times B_1 \times B_2 \to E$, is a suitable function such that one has $F(0, x^0, y^0)=\theta$, we study the following problem

$$(1) \qquad \begin{cases} F(t, x, \dot{x})=\theta \\ x(0)=x^0. \end{cases}$$

It is easy to show that (1) has a solution if it exists for

$$(2) \qquad \begin{cases} y(t)+T\left[F\left(t, x^0+\int_0^t y(s)ds, y(t)\right)\right]=y(t) \\ y(0)=y^0 \end{cases}$$

where $T, T: E \to E$, is an operator such that $T(z)=\theta \Leftrightarrow z=\theta$.

And so, we shall study the problem (2). Put $G(t, x, y)=y+T[F(t, x, y)]$, we shall consider hypotheses which guarantee the existence of a unique continuous solution for (2); moreover, we prove that the successive approximations starting from any $y \in B_2$ converge to this solution.

Similar results for the problem

$$\begin{cases} \dot{x}=f(t, x) \\ x(0)=x^0 \end{cases}$$

have been obtained by Vidossich and Kato (see [2], [6]); in order to obtain our theorem we use a different technique than the cited authors.

Finally we observe that our result is strictly more general than a previous one by Pulvirenti, as is shown by an example in *n*.3; the theorem by Pulvirenti is in [5].

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§ 2. The main result.

In this section we shall prove the following fundamental result of the paper. It is the following

Theorem 1. *Let E, B_1, B_2, I, F be as in n.1. We suppose that there exists T, T as in n.1, such that, if $G(t, x, y)=y+T[F(t, x, y)]$, one has*

(3) $$\|G(t, x, y)-y^0\|\leqslant b_2 \qquad \text{for each } (t, x, y) \in I\times B_1\times B_2,$$

(4) *for each $\varepsilon>0$ there exists $\delta>0$ and $d=d(\varepsilon)>0$, with $\lim_{\varepsilon\to 0+} d(\varepsilon)=0$, such that $|t'-t''|<\delta$, $\|x'-x''\|<\delta$, $\|y'-y''\|\leqslant d(\varepsilon)$ implies*

$$\|G(t', x', y')-G(t'', x'', y'')\|\leqslant d(\varepsilon),$$

(5) *if $J=[0, r]$, $r=\min(a, b_1/(b_2+\|y^0\|))$, there exists g, g defined on $J\times[0, 2b_1]\times[0, 2b_2]$ with values in R, such that*

(5i) *g is continuous on $J\times[0, 2b_1]\times[0, 2b_2]$,*
(5ii) *$g(t, x', y')\leqslant g(t, x'', y'')$ if $x'\leqslant x''$, $y'\leqslant y''$, for each $t \in J$,*
(5iii) *for each (t, x', y'), $(t, x'', y'') \in J\times[0, 2b_1]\times[0, 2b_2]$ we have*

$$\|G(t, x', y')-G(t, x'', y'')\|\leqslant g(t, \|x'-x''\|, \|y'-y''\|),$$

(5iv) *$v(t)=0$ is the unique nonnegative continuous function such that $v(t) \in [0, 2b_2]$ and*

$$v(t)\leqslant g\left(t, \int_0^t v(s)ds, v(t)\right) \qquad \text{for each } t \in J.$$

Then, there exists a unique continuous solution for (2) on J; and furthermore the successive approximations starting from any $y \in B_2$ converge to this solution.

Proof. Let $y \in B_2$ be; we consider the functions

$$y_n(t)=G\left(t, x^0+\int_0^t y_{n-1}(s)ds, y_{n-1}(t)\right) \qquad t \in J$$

with $y_0(t)=y$ on J, $n \in N$.

At first we show that the functions $\{y_n\}_{n\in N}$ are equicontinuous; we show this fact by using induction on n; we consider $n=1$; for fixed $\varepsilon>0$, we consider $p>0$ such that $d(p)\leqslant\varepsilon$; if $\eta>0$, $2\eta=\min(\delta, \delta/(b_2+\|y^0\|))$, $|t'-t''|<\eta$ implies that, using (4), $\|y_1(t')-y_1(t'')\|=\|G(t', x^0+t'y, y)-G(t'', x^0+t''y, y)\|\leqslant d(p)\leqslant\varepsilon$; we suppose that $|t'-t''|<\eta$ implies $\|y_m(t')-y_m(t'')\|\leqslant d(p)\leqslant\varepsilon$, for each $m\leqslant n-1$; let $m=n$ be; by using (4) we have

$$\|y_n(t')-y_n(t'')\|=\Big\|G\Big(t', x^0+\int_0^{t'} y_{n-1}(s)ds, y_{n-1}(t')\Big)$$
$$-G\Big(t'', x^0+\int_0^{t''} y_{n-1}(s)ds, y_{n-1}(t'')\Big)\Big\|\leqslant d(p)\leqslant\varepsilon;$$

then, the thesis is true.

Now, for each $n \in N$, we define a function v_n, $v_n: J\to R$, by

$$v_n(t)=\sup_{r,q\geqslant n} \|x_r(t)-x_q(t)\| \qquad t \in J.$$

Obviously, we have

(j) the mappings $\{v_n\}_{n\in N}$ are equicontinuous

(jj) $0\leqslant v_n(t)\leqslant v_{n-1}(t)$ for each $t \in J$, $n=1, 2, \cdots$.

From (jj) it follows the existence of a v, $v: J\to R$, such that $v(t)\geqslant 0$ and $v_n(t)\to v(t)$, for each $t \in J$; moreover, (j) implies that it is possible to extract a sequence $\{v_{k(n)}\}_{n\in N}$ which converges uniformly on J to a suitable continuous $\underline{v}$, $\underline{v}: J\to R$; then, $v(t)=\underline{v}(t)$; and so, by Dini's Theorem, $v_n\to v$ uniformly on J.

Since, if $r, q\geqslant n$, one has

$$\|y_{r+1}(t)-y_{q+1}(t)\|=\Big\|G\Big(t, x^0+\int_0^t y_r(s)ds, y_r(t)\Big)$$
$$-G\Big(t, x^0+\int_0^t y_q(s)ds, y_q(t)\Big)\Big\|\leqslant g\Big(t, \int_0^t v_n(s)ds, v_n(t)\Big)$$

and

$$0\leqslant v_{n+1}(t)\leqslant g\Big(t, \int_0^t v_n(s)ds, v_n(t)\Big)$$

then, if $n\to+\infty$, we have

$$0\leqslant v(t)\leqslant g\Big(t, \int_0^t v_n(s)ds, v_n(t)\Big) \qquad \text{on } J$$

and so $v(t)=0$ on J. Then, the sequence $\{y_n\}_{n\in N}$ converges to a continuous function $\bar{y}$, $\bar{y}: J\to B_2$, such that

$$\bar{y}(t)=G\Big(t, x^0+\int_0^t \bar{y}(s)ds, \bar{y}(t)\Big).$$

By virtue of (5iii) and (5iv) we can affirm that such a function is unique. Moreover, since $y \in B_2$ is arbitrary, we have proved that the iterates converge to this function $\bar{y}$. Then, we have only to show that $\bar{y}(0)=y^0$; this is true, as it is easy to prove by considering the iterates starting from y^0; in fact, in this case we have $y_n(0)=y^0$ for each

$n \in N$. The proof is complete.

Remark 1. Existence of solutions follows from a theorem proved in [1].

Remark 2. With slight changes, our argument can be used to obtain a similar result for

$$y(t)=G\left(t, \int_0^{\alpha(t)} f(t, s, y(s))ds, y(\beta(t))\right)$$

where G, f, α, β are suitable functions. Some equations of this type have been studied in some papers by Kwapisz ([3]) and Kwapisz and Turo ([4]).

§ 3. An example.

In [5] G. Pulvirenti showed the following result

Theorem 2. *Let F be a continuous function defined on $I\times B_1\times B_2$ into E. Moreover, we suppose that*

(6) *there exists three constants μ, A, L such that*

$$0\leqslant L<1,\ A>0,\ 0\neq|\mu|<\frac{1-L}{Aa}$$

for which one has

(6i) $\|y+\mu[F(t,x,y)]-y^0\|\leqslant b_2$ *for each* $(t,x,y)\in I\times B_1\times B_2$,

(6ii) $\|y'-y''+\mu[F(t,x,y')-F(t,x,y'')]\|\leqslant L\|y'-y''\|$
for each $(t,x,y'), (t,x,y'')\in I\times B_1\times B_2$,

(6iii) $\|F(t,x',y)-F(t,x'',y)\|\leqslant A\|x'-x''\|$
for each $(t,x',y), (t,x'',y)\in I\times B_1\times B_2$,

(7) *there exists $M>0$ such that*

$$\|y+\mu[F(t,x,y)]\|\leqslant M \qquad \text{for each } (t,x,y)\in I\times B_1\times B_2.$$

Then, put $J=[0,r]$, $r=\min(a, b_1/M)$ there is a unique continuous solution for (1).

It is easy to verify that the hypotheses of Theorem 2 imply our assumptions. Now, we want to show, with an example, that there are functions F which satisfy (3), (4), (5), (5i), (5ii), (5iii), (5iv) but not the assumptions of Theorem 2; more precisely, we shall construct a function which does not satisfy (6ii).

For this purpose, we put $E=C^0([0,1])$, $x^0=y^0=\theta$, $a\in\,]0,+\infty[$. If $d>0$ is such that $tgu-u<u\sqrt{u}$ in $]0,d]$, we take b_1, b_2 such that $tgb_1-b_1+b_2-b_2\sqrt{b_2}<b_2$, with $b_2\leqslant 1/8$ and $b_2\leqslant d/2$ and $b_1<\pi/2$.

Then, we consider the following F, $F\colon I\times B_1\times B_2\to E$, defined by

$$F(t, x, y)(s) \equiv h(t)(s) + tg|x(s)| - |x(s)| - y(s)\sqrt{2|y(s)|} \qquad s \in [0, 1]$$

where $h, h: I \to E$, is uniformly continuous, $h(0) = \theta$ and $\|h(t)\| \leqslant Q$, with $Q \leqslant b_2\sqrt{b_2} + b_1 - tgb_1$.

At first we show that (6ii) fails be true. We consider a function $p, p: R^+ \to R$, defined by $p(y) = \mu y\sqrt{2y} + (1-L)y$, where $\mu \in R, \mu \neq 0, L \in [0, 1[$; we observe that $p(0) = 0$ and $p'(0) = 1 - L > 0$; then, there is $y' > 0$ such that $p(y') > 0$; then, we consider $t = 0$, $x = x^0$, $y'' = y^0$, $y'(t) = y'$ for each $t \in I$; we have

$$\|y^0 - y' + \mu[F(0, x^0, y^0) - F(0, x^0, y')]\| > L\|y^0 - y'\|$$

which contradicts (6ii).

Now, we prove that the assumptions of Theorem 1 are satisfied, if T = identity on E; obviously, $F(0, x^0, y^0) = \theta$; moreover, one has

$$G(t, x, y)(s) \equiv h(t)(s) + tg|x(s)| - |x(s)| + y(s) - y(s)\sqrt{2|y(s)|} \qquad s \in [0, 1].$$

It is easy to show that (3) is true. We show (4); for this purpose we observe that

$$|(tgx' - x') - (tgx'' - x'')| \leqslant tg|x' - x''| - |x' - x''| \qquad \text{if } x', x'' \in [0, \pi/2[$$
$$|(y' - y'\sqrt{2|y'|}) - (y'' - y''\sqrt{2|y''|})| \leqslant |y' - y''| - |y' - y''|\sqrt{|y' - y''|}$$
$$\text{if } y', y'' \in [-1/8, 1/8];$$

then, we have

$$\|G(t', x', y') - G(t'', x'', y'')\| \leqslant \|h(t') - h(t'')\| + [tg\|x' - x''\| - \|x' - x''\|]$$
$$+ [\|y' - y''\| - \|y' - y''\|\sqrt{\|y' - y''\|}] \qquad (t', x', y'), (t'', x'', y'') \in I \times B_1 \times B_2$$

and so, since h and $tgx - x$ are uniformly continuous, fixed $\varepsilon > 0$, there is $\delta > 0$ and $d = d(\varepsilon)$, with $d(\varepsilon)_{\varepsilon \to 0^+} \to 0$ such that $|t' - t''| < \delta$, $\|x' - x''\| < \delta$ and $\|y' - y''\| \leqslant d(\varepsilon)$ imply $\|G(t', x', y') - G(t'', x'', y'')\| \leqslant d(\varepsilon)$.

Furthermore, if we take $g(t, x, y) = tgx - x + y - y\sqrt{y}$, we have easily (5), (5i), (5ii), (5iii); we have only to show (5iv); let v as in (5iv); then, we have

$$v(t) \leqslant tg\int_0^t v(s)ds - \int_0^t v(s)ds + v(t) - v(t)\sqrt{v(t)} \qquad t \in J;$$

If $v(t) \leqslant \int_0^t v(s)ds$, for each $t \in J$, we have $v(t) = 0$ on J by Gronwall's Lemma. We suppose that there exists $\bar{t} \in J$ such that $v(\bar{t}) > \int_0^{\bar{t}} v(s)ds$; in this case

$$0 \leqslant tgv(\bar{t}) - v(\bar{t}) - v(\bar{t})\sqrt{v(\bar{t})}, \qquad 0 < v(\bar{t}) \leqslant 2b_2$$

which is not true since $b_2 \leqslant d/2$. Then, $v(t) = 0$ on J.

The proof is complete.

References

[1] Emmanuele, G. and Ricceri, B., Sull'esistenza delle soluzioni delle equazioni differenziali ordinarie in forma implicita negli spazi di Banach, submitted.
[2] Kato, S., On the convergence of the successive approximations for nonlinear ordinary differential equations in a Banach space, Funkcial. Ekvac., **21** (1978), 43–52.
[3] Kwapisz, M., On the existence and uniqueness of solutions of a certain integral-functional equations, Annales Polon. Math., **XXXI** (1975), 23–41.
[4] Kwapisz, M. and Turo, J., Some integral-functional equations, Funkcial. Ekvac., **18** (1975), 107–162.
[5] Pulvirenti, G., Equazioni differenziali in forma implicita in uno spazio di Banach, Ann. Mat. Pura Appl., (4) **LVI** (1961), 171–192.
[6] Vidossich, G., Global convergence of successive approximations, J. Math. Anal. Appl., **45** (1974) 285–292.

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