JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **120**, 557–560 (1986)

Existence of Solutions of Ordinary Differential Equations Involving Dissipative and Compact Operators in Gelfand–Phillips Spaces*

G. EMMANUELE

Department of Mathematics, University of Catania, 95125 Catania, Italy

Submitted by V. Lakshmikantham

Received May 15, 1985

Let E be a Banach space and X be a closed, convex subset of E. If $I=[0, a)$, $a>0$, is a real interval, we consider two functions A, B from $I\times X$ into E and the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t, x(t)) + B(t, x(t)), \\ x(0) &= x_0, \end{aligned} \tag{CP}$$

where $x_0 \in X$.

The same problem with A satisfying dissipativeness conditions and B having a compact range was considered by R. H. Martin [3, 4] and E. Schecter [5, 6]. Main purpose of this note is to show that the presence in E of a property invariant under isomorphisms, the so called Gelfand–Phillips property, allows us to relaxe the compactness assumption on B considered in all of the above cited works; we underline that we have no interest in constructing approximate solutions for (CP), but only in proving that the Gelfand–Phillips property has some consequence in the study of (CP); this motivates the following assumptions

(a) *there are a subinterval I^* of I, a real sequence (ε_n) converging to zero, a sequence of absolutely continuous functions (x_n), defined on I^* with values into X, such that $\|\dot{x}_n(t)-A(t, x_n(t))-B(t, x_n(t))\| \leqslant \varepsilon_n$ for all $t\in I^*$, $n\in N$.*

* Work performed under the auspices of G.N.A.F.A. of C.N.R. and partially supported by M.P.I. of Italy.

0022-247X/86 $3.00

This happens, for example, if X is a closed ball of E centered at x_0 (under assumption (d)); for other cases, we refer the reader to [4].

(b) *E is a Gelfand–Phillips space, i.e., a Banach space such that any of its limited sets are relatively compact* (*we recall that a subset X of E is said limited iff for any sequence $(x_n^*) \subset E^*$ converging weak* to zero we have* $\lim_n \sup_{x \in X} |\langle x_n^*, x \rangle| = 0$).

For several examples of Gelfand–Phillips spaces we refer the reader to the papers [1, 2].

Concerning the function B we make the following hypothesis which is more general than compactness of the range.

(c) *$B(t, X)$ is relatively compact, for almost any $t \in I$.*

Moreover, in all of the paper we assume that A, B verify the following assumptions of Carathéodory type

(d) *A, B are measurable in $t \in I$, for any $x \in X$ and strongly–weakly continuous in x, for almost every $t \in I$; there exist two functions α, $\beta \in L^1(I)$ such that $\|A(t, x)\| \leqslant \alpha(t)$, $\|B(t, x)\| \leqslant \beta(t)$ for all $(t, x) \in I \times X$.*

Now, we show a fundamental lemma which clarifies the roles of the Gelfand–Phillips property and assumption (c):

LEMMA. *Let* (a), (b), (c), (d) *hold. If we put $y_n(t) = \int_0^t B(s, x_n(s))\, ds$, for all $t \in I^*$, then the sequence (y_n) has a uniformly converging subsequence.*

Proof. Let $\rho > 0$ be. Since $\beta \in L^1(I)$ there is $\sigma > 0$ such that for any Lebesgue measurable set J^0, $m(J^0) < \sigma$, one has $\int_{J^0} \beta(s)\, ds < \rho/2$. Let J be the set of null measure such that $B(t, X)$ is relatively compact for any $t \in I^* - J$ (from c).

Now, we consider a sequence (x_n^*) in E^* converging weak* to zero; we can suppose that $\|x_n^*\| \leqslant 1$, for the sake of brevity. For any $t \in I^* - J$ we have that the sequence $(\langle x_n^*, B(t, x_n(t)) \rangle)$ converges to zero by the compactness of $B(t, X)$; Egoroff's Theorem says that there exists a Lebesgue measurable set J^*, $J^* \subset I^* - J$, $m(J^*) < \sigma$, for which $\langle x_n^*, B(t, x_n(t)) \rangle \to 0$ uniformly on $I^* - (J \cup J^*)$. Hence, for any $n \in N$ and $t \in I^*$, we have

$$|\langle x_n^*, y_n(t) \rangle| \leqslant \int_{I^* - (J \cup J^*)} |\langle x_n^*, B(s, x_n(s)) \rangle|\, ds + \rho/2;$$

if n is sufficiently large, we have $|\langle x_n^*, y_n(t) \rangle| < \rho$. Arbitrariness of (x_n^*) in E^* implies that $(y_n(t))$ is a limited set in E, for any $t \in T^*$; the Gelfand–Phillips property of E gives its relative compactness. Since the sequence (y_n) is bounded and equicontinuous easily, Ascoli—Arzelà Theorem concludes our proof.

Once we have shown the lemma which assures the existence of a uniformly converging subsequence of (y_n), the main result follows easily with the same techniques employed by Martin in [3] (see also [4]). To obtain it we need to use comparison functions ω: $I^* \times [0, \infty) \to [0, \infty)$, with $\omega(t, 0) = 0$ for all $t \in I^*$, satisfying the assumption

(e) *ω is continuous and the unique absolutely continuous function ψ from I^* into $[0, \infty)$ such that $\dot{\psi}(t) = \omega(t, \psi(t))$, $\psi(0) = 0$, is the identically null function.*

Now we are ready to state our theorem.

THEOREM. *Let* (a), (b), (c), (d), (e) *hold. We consider the other assumptions*:

(h$_1$) $\|A(t, x) - A(t, y)\| \leqslant \omega(t, \|x - y\|)$ *for all* $(t, x), (t, y) \in I^* \times X$,

(h$_2$) *E^* is uniformly convex and* $\langle x - y, A(t, x) - A(t, y) \rangle_- \leqslant \omega(t, \|x - y\|)\|x - y\|$ *for all* $(t, x), (t, y) \in I \times X$ (*for definition of* $\langle \cdot, \cdot \rangle_-$ *we refer to* [4]),

(h$_3$) *X is open, A is uniformly continuous in $I^* \times X$ and A verifies an assumption of dissipative type as in* h$_2$).

Then, any one of the conditions (h$_1$), (h$_2$), (h$_3$) (*with conditions* (a)–(e)) *implies the existence of an absolutely continuous solution of* (CP).

Proof. Our theorem will be proved if we can extract a uniformly converging subsequence (x_{n_k}) from (x_n); indeed, its limit point will be an absolutely continuous solution of (CP) (with a standard proof) under our assumptions. This extraction will be made using the same techniques of the theorem of [3] in the case $(C5)_2$, $(C5)_3$, $(C5)_4$ as already remarked; we only observe that if (h$_2$) is verified, then E is reflexive and so it is a Gelfand–Phillips space. The proof is complete.

When (h$_2$) is verified, our result is strictly more general than the corresponding result of Martin [3]; when (h$_1$), (h$_3$) are satisfied, our results are more general than those of [3] in the case of Gelfand–Phillips spaces, but we observe that those of [3] hold in general Banach spaces: however, our theorem is more applicable than Martin's result in a situation considered in [3]; indeed, Martin furnished an example of an integro-differential equations which is not solvable using his theorem in the case $E = L^1([0, 1])$; since $L^1([0, 1])$ is a Gelfand–Phillips space, we can apply our result with a more general B than that considered in [3], if we assume hypotheses which assure that A is uniformly continuous and dissipative (use h$_3$)) or A satisfies an assumption of Lipschitzian type (use (h$_1$)). Also, we observe that the present theorem is different from the results obtained, with different techniques, by Schechter; indeed, he always assumes compactness of the range of B (an assumption more restrictive than our hypothesis (c)), but his theorems can be used in general Banach spaces.

References

1. J. Bourgain and J. Diestel, Limited operators and strict cosingularity, *Math. Nachr.* **119** (1984), 55–58.
2. J. Diestel and J. J. Uhl, Jr., Progress in vector measures, 1977–1983, *in* "Measure Theory and its Applications," Lecture Notes in Math., Vol. 1033, Springer–Verlag, Berlin, 1983.
3. R. H. Martin, Remarks on ordinary differential equations involving dissipative and compact operators, *J. London Math. Soc.* (2) **10** (1975), 61–65.
4. R. H. Martin, "Non Linear Operators and Differential Equations in Banach Spaces," Wiley, New York, 1976.
5. E. Schechter, Evolution generated by continuous dissipative plus compact operators, *Bull. London Math. Soc.* **13** (1981), 303–308.
6. E. Schechter, Evolution generated by semilinear dissipative plus compact operators, unpublished manuscript.