Math. Z. 191, 485–488 (1986)

Mathematische Zeitschrift

On the Gelfand-Phillips Property in ε-tensor Products

Giovanni Emmanuele[1] and Frank Räbiger[2]

[1] Department of Mathematics, University of Catania, Viale A. Doria 6, I-95125 Catania, Italy
[2] Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Federal Republic of Germany

In the main result of this note we show that for certain Banach spaces E and F the ε-tensor product $E\tilde{\otimes}_\varepsilon F$ inherits the Gelfand-Phillips property from E and F. In particular we obtain conditions under which spaces of vector valued continuous functions have the Gelfand-Phillips property.

1. Introduction

Let E be a Banach space. A bounded set $B \subset E$ is called a *limited set* if for every $\sigma(E', E)$-null sequence $(x'_n)_{n\in\mathbb{N}}$ in E' one has $\lim_n \sup_{x\in B} |\langle x'_n, x\rangle| = 0$. E is said to have the *Gelfand-Phillips property* if every limited set in E is relatively compact. Banach spaces having the Gelfand-Phillips property are, for example, separable Banach spaces, reflexive Banach spaces and spaces $C(K)$, where K is both compact and sequentially compact ([2], [4, p. 238]). It is easy to see that the Gelfand-Phillips property is inherited by closed subspaces.

Unexplained notation can be found in [9]. We will now introduce some basic facts concerning ε-tensor products. Most of them can be found in [9, IV.2] and [5, pp. 223]. Let E and F be Banach spaces. The ε-tensor product $E\otimes_\varepsilon F$ and the ε-norm $\|\cdot\|_\varepsilon$ on $E\otimes_\varepsilon F$ is defined as in [9, IV.2.1]. The completion of $E\otimes_\varepsilon F$ endowed with the ε-norm is denoted by $E\tilde{\otimes}_\varepsilon F$ (in the notation of [5, pp. 223] this is the same as $E\check{\otimes} F$). Let E_1 and F_1 be Banach spaces, $T\in\mathscr{L}(E, E_1)$ (=continuous linear operators from E into E_1) and $S\in\mathscr{L}(F, F_1)$. Then $T\otimes S$: $E\tilde{\otimes}_\varepsilon F \to E_1\tilde{\otimes}_\varepsilon F_1$ is the continuous linear extension of the operator from $E\otimes_\varepsilon F$ into $E_1\otimes_\varepsilon F_1$ defined by

$$\sum_{1\le i\le n} \xi_i\otimes\eta_i \to \sum_{1\le i\le n} (T\xi_i)\otimes(S\eta_i).$$

By $\operatorname{ext}(U_{E'})$ resp. $\operatorname{ext}(U_{F'})$ we denote the set of extreme points of the closed unit ball of E' resp. F'. An easy calculation shows that for $x\in E\tilde{\otimes}_\varepsilon F$ we have

$$\|x\|_\varepsilon = \sup\{|\langle x, \xi'\otimes\eta'\rangle| : \xi'\in\operatorname{ext}(U_{E'}), \eta'\in\operatorname{ext}(U_{F'})\}. \tag{1}$$

If G is a closed subspace of E, then $G\tilde{\otimes}_\varepsilon F$ is a closed subspace of $E\tilde{\otimes}_\varepsilon F$. If in addition G is complemented in E (i.e. if there exists a continuous linear projection P on E with $PE=G$), then $G\tilde{\otimes}_\varepsilon F$ is complemented in $E\tilde{\otimes}_\varepsilon F$.

2. The Gelfand-Phillips Property in Certain ε-tensor Products

Let $(E, \mathscr{T})$ be a topological vector space. A set $C\subseteq E$ is called *conditionally $\mathscr{T}$-compact* if each sequence in C contains a subsequence which is a $\mathscr{T}$-Cauchy sequence.

2.1 Theorem. *Let E and F be Banach spaces. We assume that E has the Gelfand-Phillips property and* $\operatorname{ext}(U_{F'})$ *is conditionally $\sigma(F', F)$-compact. Then $G:=E\tilde{\otimes}_\varepsilon F$ has the Gelfand-Phillips property.*

Proof. We suppose that G does not have the Gelfand-Phillips property, i.e., there exists a limited set $B\subset G$ which is not relatively compact. So we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in B which has no convergent subsequence. Without loss of generality we may assume that $(x_n)_{n\in\mathbb{N}}$ is a $\sigma(G, G')$-Cauchy sequence ([2]) and there exists $\delta>0$ such that $\|x_n-x_{n+1}\|_\varepsilon>\delta$ for each $n\in\mathbb{N}$. Thus by (1) we can find $\xi'_n\in\operatorname{ext}(U_{E'})$ and $\eta'_n\in\operatorname{ext}(U_{F'})$ such that $|\langle x_n-x_{n+1}, \xi'_n\otimes\eta'_n\rangle|>\delta$. Since $\operatorname{ext}(U_{F'})$ is conditionally $\sigma(F', F)$-compact, there exists a subsequence $(\eta'_{n_k})_{k\in\mathbb{N}}$ of $(\eta'_n)_{n\in\mathbb{N}}$ which is $\sigma(F', F)$-convergent to $\eta'\in F'$. Then $(\xi'_{n_k}\otimes(\eta'_{n_k}-\eta'))_{k\in\mathbb{N}}$ is a norm-bounded $\sigma(G', E\otimes_\varepsilon F)$-null sequence and thus a $\sigma(G', G)$-null sequence. For $\sum_{1\leqq i\leqq n}\xi_i\otimes\eta_i\in E\otimes_\varepsilon F$ we define $S(\sum_{1\leqq i\leqq n}\xi_i\otimes\eta_i):=\sum_{1\leqq i\leqq n}\langle\eta', \eta_i\rangle\,\xi_i\in E$. In this way S defines a continuous linear operator from $E\otimes_\varepsilon F$ into E. The continuous linear extension of S to G will be denoted by T. Let us mention that for every $\xi'\in E'$ we have $\xi'\circ T=\xi'\otimes\eta'$. Since continuous linear operators map limited sets into limited sets and the difference of two limited sets is limited ([2]), we obtain that $\{T(x_{n_k}-x_{n_k+1}): k\in\mathbb{N}\}\subset E$ is limited, hence relatively compact. Since $(x_{n_k}-x_{n_k+1})_{k\in\mathbb{N}}$ is a $\sigma(G, G')$-null sequence it follows that $(T(x_{n_k}-x_{n_k+1}))_{k\in\mathbb{N}}$ converges in norm to zero. Hence there is $k_0\in\mathbb{N}$ such that

$$|\langle T(x_{n_k}-x_{n_k+1}), \xi'_{n_k}\rangle|<\delta/2 \qquad \text{for all } k\geqq k_0.$$

Then for each $k\geqq k_0$ we have

$$\begin{aligned}
&|\langle x_{n_k}-x_{n_k+1}, \xi'_{n_k}\otimes(\eta'_{n_k}-\eta')\rangle|\\
&\quad\geqq|\langle x_{n_k}-x_{n_k+1}, \xi'_{n_k}\otimes\eta'_{n_k}\rangle|-|\langle x_{n_k}-x_{n_k+1}, \xi'_{n_k}\otimes\eta'\rangle|\\
&\quad=|\langle x_{n_k}-x_{n_k+1}, \xi'_{n_k}\otimes\eta'_{n_k}\rangle|-|\langle T(x_{n_k}-x_{n_k+1}), \xi'_{n_k}\rangle|\\
&\quad>\delta-\delta/2=\delta/2.
\end{aligned}$$

Therefore the set $\{x_{n_k}-x_{n_k+1}: k\in\mathbb{N}\}\subseteq B-B$ is not limited. On the other hand $B-B$ is a limited set ([2]). So we have a contradiction. □

Obviously, $\operatorname{ext}(U_{F'})$ is conditionally $\sigma(F', F)$-compact if the Banach space F has a $\sigma(F', F)$-sequentially compact dual unit ball. Examples of Banach spaces of this type are separable Banach spaces, reflexive Banach spaces, weakly compactly generated Banach spaces ([4, XIII, Thm. 4]), the duals of separable Banach spaces not containing l_1 ([4, XIII, Thm. 10]), Banach spaces whose dual space does not contain l_1 ([4, p. 226]), Banach spaces with an equivalent smooth norm ([4, p. 239]) and weak Asplund spaces ([4, p. 239]).

Since the spaces $E\tilde{\otimes}_\varepsilon F$ and $F\tilde{\otimes}_\varepsilon E$ are isomorphic ([9, p. 237]), $E\tilde{\otimes}_\varepsilon F$ has the Gelfand-Phillips property if and only if $F\tilde{\otimes}_\varepsilon E$ has the Gelfand-Phillips property. For $F=\mathbb{R}$ the Banach spaces $E\tilde{\otimes}_\varepsilon F$ and E are isomorphic. If E is a space $C(K)$, K compact, and F an arbitrary Banach space, then $E\tilde{\otimes}_\varepsilon F$ is isomorphic to the Banach space $C(K, F)$ of F-valued continuous functions on K ([9, IV.2, Example 1]). Furthermore $\operatorname{ext}(U_{C(K)'})=\{\pm\delta_x: x\in K\}$ where $\delta_x\in C(K)'$ is defined by $\langle\delta_x, f\rangle := f(x)$, $f\in C(K)$. Thus from Theorem 2.1 we obtain:

2.2 Corollary. *Let K be a compact Hausdorff space and F be a Banach space.*

(i) *If* $\operatorname{ext}(U_{F'})$ *is conditionally $\sigma(F', F)$-compact, then F has the Gelfand-Phillips property.*

(ii) *If K is sequentially compact, then $C(K)$ has the Gelfand-Phillips property* ([4, *p.* 238]).

(iii) *If $C(K)$ has the Gelfand-Phillips property and* $\operatorname{ext}(U_{F'})$ *is conditionally $\sigma(F', F)$-compact, then $C(K, F)$ has the Gelfand-Phillips property.*

(iv) *If K is sequentially compact and F has the Gelfand-Phillips property, then $C(K, F)$ has the Gelfand-Phillips property.*

Let E be a Banach space. We denote by $c_0(E)$ the Banach space of all sequences in E converging to zero ([9, IV.2, Example 2]) and by $l_1(E)$ the Banach space of all summable sequences in E ([9, p. 241]). Then $c_0(E)$ resp. $l_1(E)$ is isomorphic to $c_0\tilde{\otimes}_\varepsilon E$ resp. $l_1\tilde{\otimes}_\varepsilon E$ ([9, IV.2, Examples 2 and 4]).

2.3 Corollary. *Let E be a Banach space with the Gelfand-Phillips property. Then $c_0(E)$ and $l_1(E)$ also have the Gelfand-Phillips property.*

It is easy to see that a Banach space E has the Gelfand-Phillips property if and only if each countable limited set in E is relatively compact. Recall that a Banach space F is said to have the *separable complementation property* if every separable subspace Y of F is contained in a closed, separable, complemented subspace of F.

2.4 Corollary. *Let E be a Banach space with the Gelfand-Phillips property and F be a Banach space with the separable complementation property. Then $E\tilde{\otimes}_\varepsilon F$ has the Gelfand-Phillips property. If in addition E is isomorphic to a space $C(K)$, K compact, then $C(K, F)$ also has the Gelfand-Phillips property.*

Proof. Let $B\subset E\tilde{\otimes}_\varepsilon F$ be a countable limited set. Then there exists a closed separable subspace Y of F such that $B\subset E\tilde{\otimes}_\varepsilon Y$. Without loss of generality we can assume that Y is complemented in F. Then $E\tilde{\otimes}_\varepsilon Y$ is a complemented subspace of $E\tilde{\otimes}_\varepsilon F$. From this we obtain that B is a limited subset of $E\tilde{\otimes}_\varepsilon Y$. By Theorem 2.1 $E\tilde{\otimes}_\varepsilon Y$ has the Gelfand-Phillips property. Hence B is relatively compact in $E\tilde{\otimes}_\varepsilon Y$ and, consequently, in $E\tilde{\otimes}_\varepsilon F$. The last assertion follows from the fact that $C(K)\tilde{\otimes}_\varepsilon F$ and $C(K, F)$ are isomorphic. □

Banach spaces which have the separable complementation property are, for example, weakly compactly generated Banach spaces ([3, p. 149]), spaces $c_0(I)$ and spaces $L_p(X, \Sigma, \mu)$, $1 \leqq p < \infty$ and (X, Σ, μ) an arbitrary measure space ([7, 1.b.8] and [8, Lemma I.2]). Consequently, if E has the Gelfand-Phillips property, then for an arbitrary measure space (X, Σ, μ) the space $L_p(X, \Sigma, \mu)\tilde{\otimes}_\varepsilon E$, $1 \leqq p < \infty$, has the Gelfand-Phillips property. By [5, p. 224, Thm. 5] this generalizes Theorem 1 and 2 of [6].

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Received July 11, 1985

Added in proof. Meanwhile L. Drewnowski has shown that $E\tilde{\otimes}_\varepsilon F$ has the Gelfand-Phillips property for each pair of Banach spaces E and F with the Gelfand-Phillips property.