

THE SPACE OF COMPACT OPERATORS CONTAINS c_0 WHEN A
NONCOMPACT OPERATOR IS SUITABLY FACTORIZED*

G. EMMANUELE, Catania and K. JOHN, Praha

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In this note we generalize certain results on when the space $K(X, Y)$ of compact operators contains an isomorphic copy of the sequence space c_0 , a fact strictly connected to the nonexistence of a projection from the space $L(X, Y)$ onto the subspace $K(X, Y)$ as showed in the papers [3], [6]. One of the first results in this direction was obtained by Kalton in [7] who proved that if there is a non compact operator with a domain space X possessing an unconditional finite dimensional expansion of the identity and taking values in an arbitrary Banach space Y then c_0 embeds into $K(X, Y)$. Diestel and Morrison [1] have proved the same statement under the assumption that Y has an unconditional basis. Other results of the same nature obtained by Feder in [5], have been generalized by the authors in the recent paper [4]; in particular, it was there shown that if $L_{w^*}(X^*, Y)$ contains a noncompact operator, if the space Y has the compact approximation property and if $Y \subset Y_1$ where the space Y_1 has an unconditional expansion of the identity, then again $c_0 \subset K_{w^*}(X^*, Y)$ (here $L_{w^*}(X^*, Y)$ denotes the space of w^* - w continuous operators). Another similar result is contained in [2] where the first author proved that if there is a non compact operator factorizing through a reflexive Banach space with an unconditional basis then again c_0 embeds into $K(X, Y)$.

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In this note we show that all these results, as well as other facts from [5], actually are consequence of our Proposition 1. It describes a quite general procedure useful to construct copies of c_0 inside $K(X, Y)$ when starting from the existence of non compact operators.

We observe that the proof of our Proposition 1 below actually is a refinement of the techniques used in the previous papers; but even if not original at all, it allows us to cover (in the separable case) the old quoted results and to furnish some new facts; among them Theorem 1 is, in our opinion, the main new application.

Before finishing this Introduction we remark that in [2] and [6] it was independently shown that if a noncompact operator $T \in L(X, Y)$ factorizes through a Banach space which has an unconditional basis then $c_0 \subset K(X, Y)$; this seems to be the only old result not covered by the present ones.

Before presenting the main result we need a definition.

Definition. We shall say that $\{K_n\} \subset K(X)$ is an unconditional compact approximating sequence if $\|A_n x - x\| \rightarrow 0$ and the sum $\sum_n (A_{n+1} - A_n)x$ is weakly unconditionally Cauchy for all $x \in X$. Moreover, we shall say that such a sequence is shrinking if $\{K_n^*\}$ is an (unconditional) compact approximating sequence for X^* . A Banach space X is said to have the (shrinking) unconditional compact approximation property if there is a (shrinking) unconditional compact approximating sequence for X .

In fact the above definition is possible for nets also, but in this section **sequences** are substantial.

We shall use also the following refinement of a fact due to Kalton [7]:

Fact (K). Let $\widehat{X} \subset X^*$ be total on X , let $\widehat{Y} \subset Y^*$ be a norming subspace and suppose that the sequence $\{T_n\} \subset K(X, Y)$ has the property that $\widehat{y}(T_n x) \xrightarrow{n} 0$ for all $x \in X$ and all $\widehat{y} \in \widehat{Y}$. Suppose further that the unit ball B_X of X is $w(X, \widehat{X})$ compact, the unit ball $B_{\widehat{Y}}$ of \widehat{Y} is $w(\widehat{Y}, Y)$ compact and that the T_n 's are $w(X, \widehat{X})$ - $w(Y, \widehat{Y})$ continuous. Then $T_n \rightarrow 0$ in the weak topology of the space $L(X, Y)$.

Proof. Suppose $T \in K(X, Y)$ is $w(X, \widehat{X})$ - $w(Y, \widehat{Y})$ continuous. Then consider the compact topological space $K = B_X \times B_{\widehat{Y}}$ where on B_X we consider the $w(X, \widehat{X})$ topology and on $B_{\widehat{Y}}$ the $w(\widehat{Y}, Y)$ -topology. Let $f(x, \widehat{y}) = \widehat{y}T(x)$ define a real function on K . Using the fact that on $\overline{T(B_X)}$ the norm topology and the Hausdorff topology $w(Y, \widehat{Y})$ coincide, it is not difficult to prove that f is continuous. So we may consider T_n as continuous functions on the compact Hausdorff space K equipped with the described topology. Then our convergence assumption and the Lebesgue theorem imply that $T_n \rightarrow 0$ in the weak topology of the normed space $C(K)$ and thus also in the weak topology of the Banach space $K(X, Y)$. \square

We are now ready to give our

Proposition 1. *Let $T \in L(X, Y)$ be an operator and let $T = BA$ be a factorization of T through a Banach space E . Suppose that E is isomorphic to a subspace of a Banach space E_1 by an isomorphism J . Suppose also that \widehat{X} is a total subspace of X^* . Finally, let the following conditions (i)–(vi) be satisfied:*

- (i) *there are a Banach space Y_1 containing isomorphically Y by an isomorphism I and a bounded linear operator $\widetilde{B}: E_1 \rightarrow Y_1$, that is an extension of the operator $B: E \rightarrow Y$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E$,*
- (ii) *there are a norming subspace \widehat{Y}_1 of Y_1^* and continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that*

$$\widehat{y}_1 \left(\sum_{i=1}^n \widetilde{B}B_iJA(x) \right) \xrightarrow{n} \widehat{y}_1(\widetilde{B}JA(x)) \text{ for all } x \in X \text{ and all } \widehat{y}_1 \in \widehat{Y}_1$$

and such that $\sum B_n$ is a weakly unconditionally Cauchy (WUC) series in the space $L(E_1)$,

- (iii) *there is a sequence $\{A_n\} \subset K(E)$ of continuous operators such that, for all $x \in X$, $IBA_nA(x) \rightarrow IBA(x)$ in the $w(Y_1, \widehat{Y}_1)$ -topology,*
- (iv) *$IBA_iA: X \rightarrow Y_1$ is $w(X, \widehat{X})$ - $w(Y_1, \widehat{Y}_1)$ continuous,*
- (v) *$\widetilde{B}B_iJA: X \rightarrow Y_1$ is $w(X, \widehat{X})$ - $w(Y_1, \widehat{Y}_1)$ continuous,*
- (vi) *the unit balls of the spaces X and \widehat{Y}_1 are compact in the $w(X, \widehat{X})$ and $w(\widehat{Y}_1, Y_1)$ topologies, respectively.*

Then certain convex blocks of $\{IBA_iA\}$ are (WUC) and, in each point $x \in X$, they converge to $IT(x)$ in the $w(Y_1, \widehat{Y}_1)$ topology.

Moreover, if the operator T is not compact then the sequence space c_0 is isomorphically contained in $\overline{\text{span}}\{BA_iA\} \subset K(X, Y)$.

Proof. The conditions (ii) and (iii) give that for all $x \in X$ and all $\widehat{y}_1 \in \widehat{Y}_1$ we have

$$(1) \quad \widehat{y}_1(IBA_nA(x)) \xrightarrow{n} \widehat{y}_1(IBA(x))$$

and

$$\widehat{y}_1 \left(\sum_{i=1}^n \widetilde{B}B_iJA(x) \right) \xrightarrow{n} \widehat{y}_1(\widetilde{B}JA(x)).$$

Thus, since \widetilde{B} extends B in the sense quoted in the assumption (ii), we get easily

$$(2) \quad \widehat{y}_1(IBA_nA(x)) - \widehat{y}_1 \left(\sum_{i=1}^n \widetilde{B}B_iJA(x) \right) \xrightarrow{n} 0.$$

Now from (iv)–(v) we see that the operators

$$IBA_nA - \sum_{i=1}^n \tilde{B}B_iJA: X \rightarrow Y_1$$

are $w(X, \hat{X})$ - $w(Y_1, \hat{Y}_1)$ continuous; so we may deduce from (vi), (2) and Fact (K) that

$$U_n = IBA_nA - \sum_{i=1}^n \tilde{B}B_iJA \xrightarrow{n} 0$$

in the weak topology of the space $K(X, Y_1)$.

Now we proceed as in [9, p. 32]. Since $U_n \xrightarrow{w} 0$, we can find disjoint convex combinations (blocks) U'_j of $\{U_n\}$, such that $\sum_{j=1}^{\infty} \|U'_j\| < \infty$. Let Y'_j be the blocks of $\{Y_n\} = \{BA_nA\}$ built with the same coefficients and let us put $Z_j = Y'_{j+1} - Y'_j$. Computing, we get that

$$IZ_j = U'_{j+1} - U'_j + C'_j,$$

where C'_j 's are disjoint blocks of $\{C_n\} = \{\tilde{B}B_nJA\}$ with coefficients between 0 and 1. Now we claim that $\sum_{j=1}^{\infty} IZ_j$ is a weakly unconditionally Cauchy (WUC) series. To see this let $Z^* \in K(X, Y_1)^*$. Then we have

$$\sum_{j=1}^{\infty} |Z^*(IZ_j)| \leq 2\|Z^*\| \cdot \sum_{j=1}^{\infty} \|U'_j\| + \sum_{n=1}^{\infty} |Z^*(C_n)| < \infty$$

using the fact that $\sum_{j=1}^{\infty} C_n$ is a WUC series thanks to (ii). Indeed, (ii) means that $\|\sum_{n=1}^m \pm B_n\| \leq K$ for all m and all \pm and thus $\{\|\sum_{n=1}^m \pm C_n\|; m \in N\}$ is also bounded, meaning that $\sum C_n$ is WUC. But I is an isomorphism; so we conclude that also $\sum_{j=1}^{\infty} Z_j$ is a WUC series. Further we observe that $\sum_{j=1}^{\infty} Z_j$ is not norm convergent. Indeed, (1) may be rewritten

$$\hat{y}_1(IY_n(x)) \xrightarrow{n} \hat{y}_1(IT(x)) \quad \text{for } \hat{y}_1 \in \hat{Y}_1, x \in X$$

which implies that also for convex blocks Y'_j we have

$$(3) \quad \hat{y}_1(IY'_n(x)) \xrightarrow{n} \hat{y}_1(IT(x)).$$

Now assume that T is not compact; it easily follows that the sequence $\{Y'_n\} \subset K(X, Y)$ does not converge in the norm topology since otherwise, by (3), $\{IY'_n\}$ would converge (in the norm) to the non compact operator IT . The famous Bessaga-Pelczyński Theorem (see [8]) now ensures that a subsequence of $\{Z_j\}$ is equivalent to the unit vector basis of c_0 , which finishes the proof. \square

As a special case we might formulate

Proposition 1a. *Let $T \in L(X, Y)$ be an operator and let $T = BA$ be a factorization of T through a Banach space E . Suppose that E is isomorphic to a subspace of a Banach space E_1 by an isomorphism J and that, further, $\widehat{E} \subset E^*$ and $\widehat{E}_1 \subset E_1^*$ are subspaces such that J is $w(E, \widehat{E})$ - $w(E_1, \widehat{E}_1)$ continuous. Suppose also that \widehat{Y} is a subspace of Y^* , \widehat{X} a total subspace of X^* . Finally, let the following conditions (i)–(vi) be satisfied:*

- (i) *there are a Banach space Y_1 containing isomorphically Y by an isomorphism I , a norming subspace \widehat{Y}_1 of Y_1^* such that I is $w(Y, \widehat{Y})$ - $w(Y_1, \widehat{Y}_1)$ continuous and a $w(E_1, \widehat{E}_1)$ - $w(Y_1, \widehat{Y}_1)$ continuous bounded linear operator $\widetilde{B}: E_1 \rightarrow Y_1$, that is an extension of the operator $B: E \rightarrow Y$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E$,*
- (ii) *there are $w(E_1, \widehat{E}_1)$ -continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that $\sum_{n=1}^{\infty} \widehat{z}_1(B_n(z_1)) = \widehat{z}_1(z_1)$ for all $z_1 \in E_1$ and all $\widehat{z}_1 \in \widehat{E}_1$ and such that $\sum B_n$ is a weakly unconditionally Cauchy (WUC) series in the space $L(E_1)$,*
- (iii) *there is a sequence $\{A_n\} \subset K(E)$ of $w(E, \widehat{E})$ -continuous operators such that, for all $z \in E$, $A_n(z) \rightarrow z$ in the $w(E, \widehat{E})$ -topology,*
- (iv) *$A: X \rightarrow E$ is $w(X, \widehat{X})$ - $w(E, \widehat{E})$ continuous and bounded,*
- (v) *$B: E \rightarrow Y$ is $w(E, \widehat{E})$ - $w(Y, \widehat{Y})$ continuous and bounded,*
- (vi) *the unit balls of the spaces X and \widehat{Y}_1 are compact in the $w(X, \widehat{X})$ and $w(\widehat{Y}_1, Y_1)$ topologies, respectively.*

Then certain convex blocks of $\{IBA_iA\}$ are (WUC) and, in each point $x \in X$, they converge to $IT(x)$ in the $w(Y_1, \widehat{Y}_1)$ topology.

Moreover, if the operator T is not compact then the sequence space c_0 is isomorphically contained in $\overline{\text{span}}\{BA_iA\} \subset K(X, Y)$.

Remark 1. As we shall see below the condition (i) is usually automatically verified by considering Y embedded into an injective superspace Y_1 . A version of the Proposition where X is a quotient of an $l_1(\Gamma)$ is also possible.

Remark 2. Note that the following condition (ii)' implies the conditions (ii) in the Propositions 1 and 1a.

- (ii)' *There are $w(E_1, \widehat{E}_1)$ -continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that $\sum_{n=1}^{\infty} B_n(e_1) = e_1$ where the countable sum converges unconditionally in the norm for all $e_1 \in E_1$.*

Moreover, if $\sum_n B_n(e_1)$ converges unconditionally to e_1 for all $e_1 \in E_1$ then (ii)' together with the other assumptions of the Propositions also imply that certain

convex blocks of $\{BA_iA\}$ are, for each point $x \in X$, unconditionally converging to $T(x)$. This applies also in the Corollaries 1–4 and in the Theorem 1. Indeed, the set $\{\sum_{n=1}^m \pm B_n(e_1); m \in N\}$ is bounded for all $e_1 \in E_1$. The uniform boundedness principle then yields that the set $\{\|\sum_{n=1}^m \pm B_n\|; m \in N\}$ is bounded again.

Corollary 1([4]). *Let $T \in L_{w^*}(X^*, Y)$ be a noncompact operator. Suppose that Y has the compact approximation property and that Y is a subspace of a separable Banach space Y_1 such that Y_1 has the unconditional compact approximation property. Then $c_0 \subset K_{w^*}(X^*, Y)$.*

Proof. It is enough to choose $E = Y$ and $B = \text{Id}_Y$ in the Proposition. \square

Similarly we get the more general and new

Corollary 2. *Let $T = BA: X^* \rightarrow Y$ be a factorization of a noncompact operator T through a Banach space E such that $A: X^* \rightarrow E$ is w^* - w continuous and $B \in L(E, Y)$. Suppose that E has the compact approximation property and that E is a subspace of a separable Banach space E_1 such that E_1 has an unconditional compact approximation property. Then $c_0 \subset K(X, Y)$.*

Proof. We choose in the Proposition for Y_1 any injective Banach space containing Y , e.g. $l_\infty(B_{Y^*})$, $\widehat{Y}_1 = Y_1^*$ and $\widehat{Y} = Y^*$. \square

Similar statement may be given e.g. for the case when A is w^* - w^* continuous and B is w^* - w continuous.

Corollary 3. *Let $T = BA: X^* \rightarrow Y$ be a factorization of a noncompact operator T through a Banach space E^* such that $A: X^* \rightarrow E^*$ is w^* - w^* continuous and $B \in L(E^*, Y)$ is w^* - w continuous. Suppose that E has the compact approximation property and that E is a quotientspace of a separable Banach space E_1 such that the imbedding $J: E^* \rightarrow E_1^*$ is w^* - w^* continuous and such that E_1 has an unconditional compact approximation property. Suppose further that I is an imbedding of the space Y into the Banach space Y_1 such that the operator B has an extension to a w^* - w continuous operator $\widetilde{B}_1: E_1^* \rightarrow Y_1$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E^*$. Then $c_0 \subset K(X, Y)$.*

The next theorem is in fact a consequence of our Proposition 1. Because it has a less technical formulation, we prefer to state it separately.

Theorem 1. *Let $T \in L(X, Y)$ be a noncompact operator and let $T = BA$ be a factorization of T through a Banach space E . Suppose that*

either

- (1) E is isomorphic to a quotient space of a Banach space E_1 , the space E^* has the compact approximation property and the space E_1 has the shrinking unconditional compact approximation property

or

- (2) E is isomorphic to a subspace of a Banach space E_1 , the space E^{**} has the compact approximation property and the space E_1^* has the shrinking unconditional compact approximation property.

Then the sequence space c_0 is isomorphically contained in $K(X, Y)$.

Proof. *Case 1.* We shall apply the Proposition to the noncompact operator $T^* = A^*B^*: Y^* \rightarrow X^*$. Let $Q: E_1 \rightarrow E$ be the surjection operator. It is well known that we may choose a linear surjection $q: l_1(\Gamma) \rightarrow X$. The lifting property of $l_1(\Gamma)$ yields an operator $S: l_1(\Gamma) \rightarrow E_1$ such that $Aq = QS$. In the Proposition we may now substitute for the space Y the space X^* , for the isomorphic embedding $J: E \rightarrow E_1$ the w^* - w^* continuous embedding $Q^*: E^* \rightarrow E_1^*$, for the isomorphic embedding $I: Y \rightarrow Y_1$ the w^* - w^* continuous embedding $q^*: X^* \rightarrow l_\infty(\Gamma)$, for \tilde{B} the mapping S^* . Further we substitute $l_1(\Gamma)^{**}$ for \hat{E}_1 , X for \hat{Y} and Y for \hat{X} . Then (i)–(iv) are easily seen to be satisfied. The condition (vi) means in our case that the closed unit balls B_{Y^*} and $B_{X^{**}}$ are w^* -compact. To check (iv) it is sufficient to observe that the operators $q^*A^*A_i^*B^*$ are w^* - w continuous. But this follows immediately because these operators are w^* - w^* continuous and compact. Similarly we observe that (v) holds. Proposition 1 now gives that $c_0 \subset \overline{\text{span}}\{A^*A_i^*B^*\}$ which means that $c_0 \subset \overline{\text{span}}\{BA_iA\}$.

Case 2. In this case E^* is isomorphic to a quotient of the space E_1^* and we may apply the case (1) to the noncompact operator $T^* = A^*B^*: Y^* \rightarrow X^*$. \square

Remark 4. The case (2) in the above Theorem may also be obtained by applying the Proposition directly to the factorization of $T^{**}: X^{**} \rightarrow Y^{**}$ through the space E^{**} where $E^{**} \subset E_1^{**}$. We also embed Y into an injective Banach space.

Remark 5. If in the Theorem 1 the operator $A: X \rightarrow E$ is weakly compact or if $l_1 \not\subseteq E^*$ then the assumption concerning E in (1) may be that only E has the compact approximation property and in (2) that only E^* has the compact approximation property. Indeed, first we notice that we may assume that A^* is unconditionally convergent (otherwise A^* would fix a copy of c_0 and thus $c_0 \subset X^*$ and this in turn would imply that $c_0 \subset K(X, Y)$). If now $l_1 \not\subseteq E^*$ then by Pełczyński (see [10]) E^* has the property (V) and thus A^* is weakly compact.

The last consequence of the previous results is a slight generalization of a result due to Feder

Corollary 4 ([5]). *Let X be isomorphic to a factor space of a Banach space X_1 , X_1 having the shrinking unconditional compact approximation property. Let the space X^* have the compact approximation property and let $L(X, Y)$ contain a noncompact operator M . Then $c_0 \subset K(X, Y)$ isomorphically.*

Proof. We apply Theorem 1 (1) (after taking $X = E$, $X_1 = E_1$) to the operator $T = M \text{Id}_X$; it then yields a copy of c_0 inside $\overline{\text{span}}\{MA_i\} \subset K(Z, Y)$. \square

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Authors' addresses: G. Emmanuele, Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy, e-mail: Emmanuele@Dipmat.Unict.It; K. John, Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail Kjohn@math.cas.cz.