UNCOMPLEMENTABILITY OF SPACES OF COMPACT OPERATORS IN LARGER SPACES OF OPERATORS

G. EMMANUELE, Catania, and K. JOHN, Praha*

(Received March 4, 1994)

Summary. In the first part of the paper we prove some new result improving all those already known about the equivalence of the nonexistence of a projection (of any norm) onto the space of compact operators and the containment of c_0 in the same space of compact operators. Then we show several results implying that the space of compact operators is uncomplemented by norm one projections in larger spaces of operators. The paper ends with a list of questions naturally rising from old results and the results in the paper.

Keywords: spaces of linear and compact operators, non existence of projections, copies of c_0, Approximation properties, non existence of norm one projection, Hahn-Banach extensions

MSC 1991: 46A32, 46B20, 46H10, 46B99, 46B25

1. INTRODUCTION

Let $K(X,Y)$, $W(X,Y)$ and $L(X,Y)$ denote respectively the Banach spaces of linear and compact, weakly compact and bounded operators from the Banach space X to the Banach space Y, equipped with the sup norm. In this paper we consider the long standing question: (Q)*Is it true that, for all X and Y, either $K(X,Y) = L(X,Y)$ or $K(X,Y)$ is an uncomplemented subspace of $L(X,Y)$?*

This question has been treated in several papers before (see [Ka1], [F1], [F2], [E3], [E4], [J1], [J2]) where the authors gave conditions for (Q) to possess a positive answer; in particular in [E3] and [J1] the authors showed independently that if $K(X,Y)$

* This work was completed while the second-named author was visiting the University of Catania, thanks to a financial support by C.N.R. of Italy; the work of the first-named author was supported by M.U.R.S.T. of Italy (40 %,1992)

contains an isomorphic copy of c_0 then it is uncomplemented in $L(X,Y)$ and that c_0 is contained in $K(X,Y)$ if there is a non compact operator from X to Y that factorizes through a space with an (uncountable or countable) unconditional finite dimensional expansion of the identity. And as far as we know these are the most general results existing about (Q), also because it seems that up to know only two spaces for which c_0 does not embed into $K(X,Y)$ and further $K(X,Y) \neq L(X,Y)$ are known (see [E3], [E5]); they are two $\mathcal{L}_\infty$-spaces constructed by Bourgain and Delbaen ([BD]): one does not contain copies of c_0 and has a dual isomorphic to l_1, the other has Schur property and a non separable dual.

In passing we observe that starting from the existence of these two spaces it is not difficult to show, thanks to the results in [E3], [E4] and [E5], that

i) if X is a Banach space without copies of c_0 inside and with dual isomorphic to l_1, then $K(c_0(X), X)$, different from $L(c_0(X), X)$, does not contain copies of c_0,

ii) if X is a Banach space without copies of c_0 inside and with dual isomorphic to l_1, then $K(X, L_p(X))$, different from $L(X, L_p(X))$, $1 \leqslant p < \infty$, does not contain copies of c_0,

iii) if X is a Banach space with Schur property without copies of c_0 inside the dual, then $K(c_0(X), X)$, d,ifferent from $L(c_0(X), X)$, does not contain copies of c_0

iv) if X is a Banach space with Schur property without copies of c_0 inside the dual, then $K(C(\Omega, X), X)$, different from $L(C(\Omega, X), X)$, Ω a compact Hausdorff space, does not contain copies of c_0,

v) if X is a Banach space with Schur property without copies of c_0 inside the dual, then $K(L_p(X), X)$, different from $L(L_p(X), X)$, $1 < p < \infty$, does not contain copies of c_0,

this way getting more spaces for which no answer to question (Q) is so far known.

In the first part of the paper we study the equivalence of the following five conditions (already considered by a number of authors, see [Ka1], [F], [E3], [E4], [J1]):

a) *$L_{w^*}(X^*, Y) \neq K_{w^*}(X^*, Y)$ (resp. $L(X,Y) \neq K(X,Y)$ or $W(X,Y) \neq K(X,Y)$).*

b) *c_0 embeds in $K_{w^*}(X^*, Y)$ and X and Y do not have Schur property (resp. c_0 embeds in $K(X,Y)$ or c_0 embeds in $K(X,Y)$ and X^* and Y do not have Schur property).*

c) *c_0 embeds in $L_{w^*}(X^*, Y)$ and X and Y do not have Schur property (resp. c_0 embeds in $L(X,Y)$ or c_0 embeds in $L(X,Y)$ and X^* and Y do not have Schur property).*

d) *l_∞ embeds in $L_{w^*}(X^*, Y)$ and X and Y do not have Schur property (resp. l_∞ embeds in $L(X,Y)$ or l_∞ embeds in $W(X,Y)$ and X^* and Y do not have Schur property).*

e) *$K_{w^*}(X^*,Y)$ is not complemented in $L_{w^*}(X^*,Y)$ (resp. $K(X,Y)$ is not complemented in $L(X,Y)$ or in $W(X,Y)$).* (by $K_{w^*}(X^*,Y)$ and by $L_{w^*}(X^*,Y)$ we denote, respectively, the Banach spaces of all weak*-weak continuous compact and weak*-weak continuous bounded operators from X^* to Y, equipped with the sup norm). In this second section we in particular strongly generalize results by Feder ([F1]).

The second part is devoted to a study of the other question, naturally related to question (Q), of the existence of norm one projection onto the space of compact operators. We observe, for instance, that $K(X)$ is not norm one complemented in $L(X)$ if X has the Radon-Nikodym property; this result also applies to both the Bourgain-Delbaen spaces ([BD]) already quoted. Furthermore, we show that $K(X,Y)$ is not complemented by a norm one projection if, for instance, any of a), b) or c) holds

j) *the norm of X^* is Gauteaux differentiable on X^* and the norm of Y is Frechet differentiable on a dense subset,*

jj) *X and Y^* have the Radon-Nikodym property,*

jjj) *either X^{**} or Y^* has the Kadec-Klee property for nets.*

Here results by Lima ([Li2]) and Lima, Oja, Rao, Werner ([LORW]) are essentially used.

At the end of the paper we quote several questions naturally rising from old results about question (Q) and the present results.

2. Uncomplementability by arbitrary projections

In this section we present some sufficient conditions implying uncomplementability by arbitrary projections. More precisely, we show that the five facts a)–e) from the Introduction are (sometimes) equivalent under additional assumptions as done in several other papers ([Ka1], [F1], [E3], [E4], [J1]). As it follows from results in [Ka1], [F1], [F2], [E3], [E4], [J1] the following implications are true for arbitrary Banach spaces X and Y: b) $\Rightarrow$ d), b) $\Rightarrow$ e), d) $\Rightarrow$ a); since e) $\Rightarrow$ a) and b) $\Rightarrow$ c) are obvious, in order to show the required equivalences it is enough to prove that a) $\Rightarrow$ b).

In particular our attention is directed to Feder's paper [F1] in which the author considers the spaces $L(X,Y)$ and $K(X,Y)$ and shows that, under certain hypotheses on X and Y, a) $\Rightarrow$ b). It is not difficult to realize (see below) that Feder's hypotheses in his Theorem 4 ((1) and (3)) imply that $L(X,Y)$ actually coincides with $W(X,Y)$ if one assumes that c_0 does not embed into X^* and Y (but if c_0 embeds into either X^* or Y, then b) is trivially true under no other assumption on X^* and Y); this means that Feder's assumption $L(X,Y) \neq K(X,Y)$ actually implies that there is a weakly compact operator from X to Y that is not compact; using this remark as a starting point we are then able to extend Feder's results mainly by generalizing his assumptions on X; in fact our proofs are standard (see the papers [Ka1], [F1], [F2], [E3], [E4], [J1]), but we take great advantage of the weak compactness of the (non compact) operator which existence is guaranteed by a), a fact not considered in previous papers. We also observe that most of our results are stated (as in [E4]) for classes of operators more general than those considered in the other quoted papers. Furthermore, the paper [F1] contains one more result (Theorem 4 (2)) in which it is assumed that X is a weakly compactly generated Banach space; we are able to dispense at all with this assumption. Sometime, we shall also use an approximation property type assumption that is strictly more general than those used in [F1], thanks to a recent example contained in the paper [Wi].

In order to start we need the following definitions.

Definition 2.1, [LTI], [LTII]. A Banach space X is said to have the compact approximation property (c.a.p.) if there is a bounded net (A_α) contained in $K(X)$ such that the limit relationship $\|A_\alpha(x)-x\| \longrightarrow 0$ holds, for each $x \in X$. If moreover $\|A_\alpha^*(x^*) - x^*\| \longrightarrow 0$ for each $x^* \in X^*$ then we say that X has the shrinking (c.a.p.) (in symbols (s.c.a.p.)) It is clear that if X is separable (or if X^* is separable), we can choose a sequence instead of a net in Definition 2.1.

Definition 2.2. We say that a Banach space X has an unconditional compact expansion of the identity (u.c.e.i.) if there is a bounded sequence $(A_n) \subset K(X)$ such that $\sum A_n(x) = x$ unconditionally for each $x \in X$. It is clear that each space with an (u.c.e.i.) is separable. In passing we observe that the existence of a (u.c.e.i) is implied by a recently considered approximation property, called (UKAP) in [Ka2], that is in turn consequence of the fact that $K(X)$ is M-ideal in $L(X)$.

Since X (resp. X^*) and Y are closed subspaces of $K_{w^*}(X^*,Y)$ (resp. of $K(X,Y)$) we may assume (and will do) that c_0 does not embed into X (resp. X^*) and Y in the proof of all statements below, because otherwise c_0 would embed trivially into the considered spaces of compact operators. We are now ready for proving the first result.

Theorem 2.3. *Let X be an arbitrary Banach space and Y be a Banach space with the (c.a.p.). Let us assume Y is a closed subspace of a Banach space Z possessing an (u.c.e.i.). If $L_{w^*}(X^*,Y) \neq K_{w^*}(X^*,Y)$, then c_0 embeds in $K_{w^*}(X^*,Y)$.*

Proof. Let $T \in L_{w^*}(X^*,Y) \setminus K_{w^*}(X^*,Y)$ be. Since Y is separable and has the (c.a.p.), there are $A_n \in K(Y)$, $n \in N$, such that $A_n(y) \longrightarrow y$ for all $y \in Y$ and $A_n^*(y^*) \xrightarrow{w^*} y^*$, for all $y^* \in Y^*$. Since T is w*-w continuous we have $T^*(A_n^*(y^*)) \xrightarrow{w} T^*(y^*)$ in X, for all $y^* \in Y^*$, which clearly means that $T^*(A_n^*(y^*))(x^*) \longrightarrow T^*(y^*)(x^*)$ for all $x^* \in X^*$, $y^* \in Y^*$. Let j denote the isomorphic embedding of Y into Z; hence j^* maps Z^* into Y^* and so $j^*(z^*) \in Y^*$ from which we get the following limit relationship $T^*(A_n^* j^*(z^*))(x^*) \longrightarrow T^*(j^*(z^*))(x^*)$ for all $z^* \in Z^*$ and $x^* \in X^*$. Since Z has an (u.c.e.i.), there is $\sum B_i$ in $K(Z)$ such that, for all $x^* \in X^*$, we have $\sum B_i(jT(x^*)) = jT(x^*)$ unconditionally in Z. This implies that the series $\sum B_i jT$ is weakly unconditionally converging (see [F1], Introduction) in $K_{w^*}(X^*, Z)$, but not unconditionally converging. Using the w*-w continuity of T as above, we get the following limit relationship

$$T^* j^* [\sum_{i=1}^{n} B_i^*(z^*)](x^*) \longrightarrow T^* j^*(z^*)(x^*)$$

for all $z^* \in Z^*$, $x^* \in X^*$, from which it follows that

$$j A_n T - \sum_{i=1}^{n} B_i j T \xrightarrow{w} 0$$

in $K_{w^*}(X^*, Z)$. The final part of our proof (which consists in showing that suitable convex combinations of the $A_n T$'s can be taken to get a weakly unconditionally converging series in $K_{w^*}(X^*, Y)$ that is not unconditionally converging) is equal to the proof of Proposition 1.c.3 in [LTII, p. 32] and so it will be not included here. We are done. □

In the first Corollary of Theorem 2.3 by Σ we denote a σ-algebra admitting a nonzero atomless finite positive measure and by $ca(\Sigma, Y)$ (resp. $cca(\Sigma, Y)$) the Banach space of all countably additive measures (resp. compact countably additive measures) from Σ to Y equipped with the semivariation norm.

Corollary 2.4. *Let Y be as in Theorem 2.3. If $ca(\Sigma, Y) \neq cca(\Sigma, Y)$, then c_0 embeds (complementably,* see [E4]*) in $cca(\Sigma, Y)$. Thus $cca(\Sigma, Y)$ is uncomplemented in $ca(\Sigma, Y)$ (*see [E4]*)*

Proof. It is well known that $ca(\Sigma, Y)$ (resp. $cca(\Sigma, Y)$) is isometrically isomorphic with $L_{w^*}((ca(\Sigma))^*, Y)$ (resp. $K_{w^*}((ca(\Sigma))^*, Y)$), (see [Ru]). □

Corollary 2.4 is also an improvement of Corollary 14 in [E4].

Corollary 2.5. *Let X be an arbitrary Banach space and Y be a Banach space with the (c.a.p.). Let us assume Y is a closed subspace of a Banach space Z possessing an (u.c.e.i.). If $W(X, Y) \neq K(X, Y)$, then c_0 embeds in $K(X, Y)$.*

Proof. It is enough to observe that $W(X, Y)$ is isometrically isomorphic to $L_{w^*}(X^{**}, Y)$ and $K(X, Y)$ to $K_{w^*}(X^{**}, Y)$ (see [Ru]). □

We do not know if the thesis of Corollary 2.5 holds true if we drop the assumption of the existence of an (u.c.e.i.); from the results in the next section 3, with Y separable, we get that there is no norm one projection of $W(X,Y)$ onto $K(X,Y)$. The main problem here seems to be that we do not know of any pair of Banach spaces X and Y for which $W(X,Y) \neq K(X,Y)$ and c_0 doesn't embed into $K(X,Y)$.

We recall that a Banach space X is said to be a *Grothendieck space* if each operator from X to a separable space is weakly compact ([DU]) and that X is said to possess *property (V) of Pelczynski* if each unconditionally converging operator T defined on X is weakly compact ([P]).

Corollary 2.6. *Let X be a Grothendieck space or a Banach space with property (V) of Pelczynski and Y, Z be as in Theorem 2.3. If $L(X,Y) \neq K(X,Y)$ then c_0 embeds in $K(X,Y)$.*

Proof. If Y contains a copy of c_0 we are trivially done. Otherwise, it is enough to observe that $L(X,Y) = W(X,Y)$. This is true since Y is separable and X is a Grothendieck space or because we are assuming that Y does not contain copies of c_0 and X has property (V) of Pelczynski. Then we apply Corollary 2.5. □

This Corollary 2.6 improves Theorem 4, (1) from [F1] obtained under the assumptions "X is reflexive and Y has the (b.a.p.) and is a closed subspace of a Banach space Z with an unconditional basis".

Corollary 2.7. *Let X be a separably complemented Banach space (*see [B], p. 304, *for instance). If for each separable subspace Y_1 of Y there is a Banach space Z_1 such that Y_1 and Z_1 are as Y and Z in Corollary* 2.5, *then the same conclusion of Corollary* 2.5 *holds true.*

Proof. Let $T \in W(X,Y) \setminus K(X,Y)$ be. There is a separable subspace X_1 of X complemented in X with T restricted to X_1 not compact.Let us put $Y_1 = \overline{\text{span}}(T(X_1))$. It is now enough to apply Corollary 2.5 to $W(X_1,Y_1)$ and $K(X_1,Y_1)$ and to observe that $K(X_1,Y_1)$ is a closed subspace $K(X,Y)$. We are done. □

It would be interesting to know if the assumption "X is separably complemented" in Corollary 2.7 can be eliminated.

Corollary 2.5 can be also utilized in a different way as it follows: there are a lot of results concerning isomorphic properties of Banach spaces in spaces of operators in which an assumption of coincidence of the spaces $K(X,Y)$ and $W(X,Y)$ is done; Corollary 2.5 allows us to state that sometimes this assumption is even necessary for the validity of those results. In [Le] the author showed that the equality $K(X,Y) = W(X,Y)$ is sufficient (and necessary, too, in case one of the two spaces X and Y has the metric approximation property) for $K(X,Y)$ to be weakly sequentially complete. We have the following

Corollary 2.8. *Let X, Y, Z be as in Corollary 2.5. If $K(X,Y)$ is weakly sequentially complete, then $K(X,Y) = W(X,Y)$.*

Similarly, from Corollary 2.5 we get the following partial converse to Theorem 5 in [E1].

Corollary 2.9. *Let X be a Grothendieck space and Y be a reflexive Banach space such that Y^* has the (c.a.p.) and is a closed subspace of a Banach space Z with an (u.c.e.i.). If $X \otimes_\pi Y$ is a Grothendieck space, then $K(X,Y^*) = W(X,Y^*)$.*

An important consequence of Theorem 2.3 is the following result.

Theorem 2.10. *Let X^* be a closed subspace of a Banach space Z with an (u.c.e.i.). If X^* has the (c.a.p.) and $W(X,Y) \neq K(X,Y)$, then c_0 embeds in $K(X,Y)$, regardless of the nature of Y.*

Proof. The result follows by Theorem 2.3 just taking adjoints. □

First of all we observe that Theorem 2.10 is an immediate improvement of Corollary 9 and Corollary 11, (2) in [E4]. Moreover, as corollary of it we present a result due to Feder ([F1], Theorem 4, (3)).

Corollary 2.11 [F1]. *Let X be a Banach space that is a quotient of a Banach space E with a shrinking unconditional basis and X^* has the (b.a.p.). Then $L(X,Y) \neq K(X,Y)$ implies that c_0 embeds in $K(X,Y)$.*

Proof. If c_0 embeds into either X^* or Y we are trivially done. So let us suppose that c_0 doesn't live into X^* and Y. Since E has a shrinking unconditional basis, then E has property (u) of Pelczynski and it is not allowed to contain copies of l_1 (both of these facts can be found in [LTI], for instance). Hence it has property (V) of Pelczynski that is inherited by quotients. So X too enjoys property (V) of Pelczynski. Again we may assume that Y does not contain c_0 and thus each operator T from X to Y must be weakly compact ([P]). Since the other hypotheses of Theorem 2.9 are easily verified we are done. □

The proof of Corollary 2.11 shows that Theorem 2.10 improves greatly the result due to Feder ([F1], Theorem 4, (3)) and make clearer our remarks at the beginning of the section on the relationships between the present results and those in the paper [F1]. Moreover, Theorem 2.10 is easily seen to be an improvement over Theorem 6 in [Ka1].

The last result we present is about the spaces $L(X,Y)$ and $K(X,Y)$.

Theorem 2.12. *Let X be arbitrary and Y be a Banach space contained (as a closed subspace) in a Banach space Z with an (u.c.e.i.) $\sum A_n$ such that $\sum A_n^*(z^*) = z^*$ in the weak topology of X^* for all $z^* \in Z^*$. If either X^* or Y^* has the (b.a.p.) and $L(X,Y) \neq K(X,Y)$, then c_0 embeds in $K(X,Y)$.*

Proof. Let us first assume that X^* has the (b.a.p.) (we proceed similarly if Y^* has the (b.a.p.)) and let $T \in L(X,Y) \setminus K(X,Y)$ be. Since Y^* and $T^*(Y^*)$ are separable we can find a sequence $(B_n) \subset K(X)$ such that $(B_n^* T^*)(y^*) \longrightarrow T^*(y^*)$ in the norm of X^*, for all $y^* \in Y^*$ (using Lemma 3.1 in [JRZ]) from which it follows that, for all $z^* \in Z^*$, $x^{**} \in X^{**}$,

$$B_n^* T^*(j^*(z^*))(x^{**}) \longrightarrow T^*(j^*(z^*))(x^{**})$$

where j still denotes the isomorphic embedding of Y into Z. Now, the same proof of Theorem 2.3 works (using the fact that $\sum A_i^*(z^*) \overset{w}{=} z^*$ for all $z^* \in Z^*$ instead of the w*-w continuity of T) to get the existence of a weakly unconditionally converging series in $K(X,Y)$, that is not unconditionally converging. We are done.

This last Theorem 2.12 improves, as remarked at the beginning of this section, a result due to Feder ([F1], Theorem 4, (2)) obtained under the assumptions "X is weakly compactly generated, Y is a closed subspace of a Banach space Z with a shrinking unconditional basis and either X^* or Y^* has the (b.a.p.)" as well as Corollary 11, (1), in [E4]. We do not know if we can drop at all the assumption of the existence of an (u.c.e.i); the best consequence we are able to get without this assumption is that $K(X,Y)$ is not norm one complemented in $L(X,Y)$, as Proposition 3.1 in the next section will soon show. But this may perhaps suggest that some unconditionality assumption must be considered if one wants to get uncomplementability.

3. Uncomplementability by norm one projections

This section is devoted to state results proving that in certain cases there exist no projections $P\colon L(X,Y) \longrightarrow K(X,Y)$ with $\|P\| = 1$.

The general idea we use to prove this is the following: supposing that P is a projection from $L(X,Y)$ onto $K(X,Y)$, under appropriate assumptions we are sometimes able to show that $P^*(x^* \otimes x) = x^* \otimes x$ for "sufficiently many" $x^* \otimes x$ in $L(X,Y)^*$; this means that $P^*(L(X,Y)^*)$ is total and hence P must be one to one, a contradiction if $K(X,Y) \neq L(X,Y)$. More generally: let $H = \overline{\operatorname{span}}(K(X,Y) \cup \{T\})$, where $T \in L(X,Y) \setminus K(X,Y)$. Suppose that "sufficiently many" elements of the form

$\varphi = x^* \otimes x \in K(X,Y)^*$ have unique Hahn-Banach extension to all of $H \subset L(X,Y)$. If P were a norm one projection of H onto $K(X,Y)$, then $\varphi \circ P$ should be equal to this unique extension $\hat{\varphi}$. From this fact it follows that

$$\varphi PT = \hat{\varphi}T, \quad \text{i.e.} \quad x^*PT(x) = x^*T(x)$$

for "sufficiently many" x^* and x, allowing to conclude that $PT = T$, a contradiction.

In passing we observe that if each element in $(K(X))^*$ can be extended in a unique norm preserving way to an element of $(L(X))^*$, then the space X must possess the (c.a.p.). Indeed, let us suppose that each element in $(K(X))^*$ can be extended in a unique norm preserving way to an element of $(L(X))^*$. Then if $\varphi \in K(X)^*$ and $\hat{\varphi} \in L(X)^*$ is its unique Hahn-Banach extension, for each $T \in L(X)$ we can define $\psi(T) \in K(X)^{**}$ by putting $\psi(T)(\varphi) = \hat{\varphi}(T)$ (to show the additivity of $\psi(T)$ use condition (3) in Theorem 4.1 from [Li1]). Furthermore, it is easily seen that it is an isometry, from which it follows that $L(X)$ can be identified with a closed subspace of $K(X)^{**}$. Hence, there is a net (A_α), contained in the unit ball of $K(X)$, that converges to Id_X in the w*-topology of $K(X)^{**}$. Corollary VI.1.5 in [DS] allows us to conclude. We are done. □

We recall that in [E3] and [J1] it was shown that if c_0 is embedded in $K(X,Y)$, then we have no projection (of any norm) from $L(X,Y)$ onto $K(X,Y)$; the results of this section are not included in it (neither in those of the previous Section 2), as we shall see after getting the first result of the section.

Proposition 3.1. *Let X be a Banach space with the Radon-Nikodym property. Then there is no norm one projection of $L(X)$ onto $K(X)$.*

Proof. Let $H = \overline{\text{span}}(K(X) \cup Id_X)$. Let x be a strongly exposed point of the unit ball of X and $x^* \in X^*$ be the exposing functional; put $\varphi = x^* \otimes x$. Then by [Li2], Lemma 12, φ has a unique norm one extension $\hat{\varphi}$ to all of H. Thus if P were a norm one projection from $L(X)$ onto $K(X)$, then $\varphi P = \hat{\varphi}$ and specially $(\varphi \circ P)Id_X = \hat{\varphi}(Id_X)$. Hence $x^*P(x) = x^*(x)$. Since x is exposed by x^* we can conlude that $Px = x$. Because the unit ball of X is the closed, convex hull of its strongly exposed points, we get $PId_X = Id_X$, a contradiction. □

As remarked at the beginning of the section, the result of this section are not contained in the old papers on the same subject; Proposition 3.1 indeed applies to the Bourgain-Delbaen spaces.

In the next results we shall consider the case of a domain space not necessarily equal to the range space.

Proposition 3.2. *Let X, Y be two Banach spaces such that the norm of X^* is Gauteaux differentiable on X^* and the norm of Y is Fréchet differentiable on a dense subset. Then there is no norm one projection of $L(X,Y)$ onto $K(X,Y)$ provided that $L(X,Y) \neq K(X,Y)$.*

Proof. The proof follows that one of Theorem 8.3 in [GKS]. Let x^* and y be points of Gauteaux and Fréchet smoothness of the norm of X^* and Y, respectively, both with norm one. Let $x^{**} \in S(X^{**})$, $y^* \in S(Y^*)$ be the unique tangents at x^* and y. Then the operator $A = x^* \otimes y \in K(X,Y)$ is a point of Gauteaux smoothness in $L(X,Y)$ (see the proof of Theorem 8.3,(b), in [GKS]); thus for any scalar α we have, with $T \in L(X,Y)$,

$$\|A + \alpha T\| = 1 + \alpha \langle y^* \otimes x^{**}, T\rangle + o(|\alpha|).$$

Hence

$$\|A + \alpha PT\| \leqslant 1 + \alpha \langle y^* \otimes x^{**}, T\rangle + o(|\alpha|).$$

In particular

$$\langle y^* \otimes x^{**}, A + \alpha PT\rangle \leqslant 1 + \alpha \langle y^* \otimes x^{**}, T\rangle + o(|\alpha|).$$

Hence

$$(1). \qquad \langle y^* \otimes x^{**}, PT\rangle = \langle y^* \otimes x^{**}, T\rangle$$

Now, since the y's run through a dense subset of Fréchet smoothness, the $y^{*\prime}$s are dense in Y^* and similarly the $x^{**\prime}$s form a dense subset of X^{**}, by Bishop-Phelps Theorem. Thus (1) implies easily that $PT = T$. If we choose $T \notin K(X,Y)$ then we have a contradiction. □

Using Lemma 11 in [Li2] we can also prove the following

Proposition 3.3. *Let X and Y^* have the Radon-Nikodym property. Then $K(X,Y)$ is not complemented by a norm one projection in $W(X,Y)$ (resp. in $L(X,Y)$), provided $W(X,Y) \neq K(X,Y)$ (resp. $L(X,Y) \neq K(X,Y)$).*

Proof. If x is a denting point of the unit ball of X and y^* a w*-denting point of the dual unit ball of Y, Lemma 11 in [Li2] implies that the functional $x \otimes y^*$ has a unique Hahn-Banach extension to all of $L(X,Y)$ (and hence of $W(X,Y)$). On the other hand, since X has the Radon-Nikodym property, then its closed unit ball is the closed convex hull of its denting points and since Y^* has the Radon-Nikodym property, too, its closed unit ball is the w*-closed convex hull of its w*-denting points (see [B]); this means that "sufficiently many" elements in the unit ball of $K(X,Y)^*$ have unique norm preserving extension to all of $W(X,Y)$ (or $L(X,Y)$). As already remarked at the beginning of the section, this is sufficient to get our thesis. □

If we consider a stronger assumption on either X^{**} or Y^* (namely the Kadec-Klee property for nets, [NP]) we can drop any restriction on the other space.

Proposition 3.4. *Let X, Y be two Banach spaces such that either X^{**} or Y^* has the Kadec-Klee property for nets, i.e. for all (f_α), f in X^{**} (or Y^*) such that $f_\alpha \xrightarrow{w^*} f$ and $\|f_\alpha\| \longrightarrow \|f\|$, one has $f_\alpha \longrightarrow f$ in the norm topology. Then $K(X,Y)$ is not complemented in $W(X,Y)$ (resp. in $L(X,Y)$) by a norm one projection, provided $W(X,Y) \neq K(X,Y)$ (resp. $L(X,Y) \neq K(X,Y)$).*

Proof. The proof is the same as that of Proposition 3.5; it is enough to use Corollary 3.10 in [LORW]. □

Other results on uncomplementability by norm one projection of $K(X,Y)$ in $W(X,Y)$ and of $K_{w^*}(X^*,Y)$ in $L_{w^*}(X^*,Y)$ can be obtained similarly.

Proposition 3.5. *Let X^* and Y^* have the Radon-Nikodym property. Then $K_{w^*}(X^*,Y)$ is not complemented by a norm one projection in $L_{w^*}(X^*,Y)$, provided $L_{w^*}(X^*,Y) \neq K_{w^*}(X^*,Y)$.*

Proof. The proof is similar to those of the previous results. It is enough to consider elements of the type $x^* \otimes y^* \in K_{w^*}(X^*,Y)$ with x^* and y^* w*-strongly exposed points of the unit ball of X^* and of Y^*, respectively, and to remark that the same proof of Lemma 11 in [Li2] gives that such elements have unique norm preserving extension to all of $L_{w^*}(X^*,Y)$. □

All of the above results about uncomplementability by norm one projections may be generalized as it follows.

Proposition 3.6. *Let X, Y be arbitrary Banach spaces and let X_0 (resp. Y_0) be a norm one complemented subspace of X (resp. Y) such that $K(X_0,Y_0)$ is not norm one complemented in $L(X_0,Y_0)$ or $W(X_0,Y_0)$ (see previous results in the section). Then $K(X,Y)$ is not norm one complemented in $L(X,Y)$ or $W(X,Y)$.*

Proof. Let us denote by P_X (resp. P_Y) the norm one projection of X (resp. Y) onto X_0 (resp. Y_0). If P were a norm one projection of $L(X,Y)$ or $W(X,Y)$ onto $K(X,Y)$, then we could define a norm one projection $\tilde{P}$ from $L(X_0,Y_0)$ or $W(X_0,Y_0)$ onto $K(X_0,Y_0)$ by putting $\tilde{P}(T) = [P_Y \circ P(T \circ P_X)]_{|X_0}$ for each $T \in L(X_0,Y_0)$. This fact would contradict our hypothesis. We are done. □

At the end of this section we remark that in the paper [Jo] it is proved that if X or Y is a Banach space with the metric (c.a.p.), then there is an isometry Ψ of $L(X,Y)$ into $K(X,Y)^{**}$ so that Ψ restricted to $K(X,Y)$ is the canonical embedding. This allows us to show the following

Proposition 3.7. *Let X or Y have the metric (c.a.p.) and X be an arbitrary Banach space. Let us suppose that the pair X, Y verifies the assumptions of one of the Propositions in this section on the nonexistence of norm one projections. Then $K(X,Y)$ is not a dual space.*

Proof. If $K(X,Y)$ were a dual space it would be complemented by a norm one projection in its bidual and hence (thanks to the previous remark) in $L(X,Y)$ or $W(X,Y)$, which is forbidden by the results of this section. □

Remark 3.8. The "smallest" ideal we consider in this paper is $K(X,Y)$, but it is not difficult to see that the results in this section (as well as the main results in [E3] and [J1]) remain valid if we change $K(X,Y)$ to $\overline{F}(X,Y)$, i.e. the uniform closure of finite dimensional operators.

4. Some final questions

In the light of the results in this paper as well as in previous papers (see [Ka1], [E3], [E4], [E5], [F1], [F2], [J1], [J2]) it seems quite natural to put the following questions

Question 4.1. Is $K(X,Y)$ complemented or not in $L(X,Y)$ in some of the known cases (see Introduction) for which c_0 does not embed into $K(X,Y)$ and $K(X,Y) \neq L(X,Y)$?

Question 4.2. Does a pair of reflexive Banach spaces X, Y exist so that c_0 does not embed into $K(X,Y)$ and $K(X,Y) \neq L(X,Y)$? In case of positive answer what about Question 4.1 for such spaces?

Question 4.3. If X, Y are spaces to which Question 4.1 is applicable, do there exist subspaces X_0 of X and Y_0 of Y for which $K(X/X_0,Y_0) \neq L(X/X_0,Y_0)$? In case of positive answer what about Question 4.1 for such spaces (we recall that $K(X/X_0,Y_0) \subset K(X,Y)$ so that c_0 does not embed in $K(X/X_0,Y_0)$)?

More generally we can ask

Question 4.4. Is it true that $K(X,Y)$ is uncomplemented in $L(X,Y)$ if and only if it contains a copy of c_0?

Question 4.5 [E5]. Do two Banach spaces X, Y exist so that c_0 does not embed into $K(X,Y)$ and however $K(X,Y) \neq W(X,Y)$?

Question 4.6. Is it true that $K(X,Y)$ is never norm one complemented in $L(X,Y)$?

Question 4.7. Is it true that c_0 embeds into $K(X,Y)$ if there is a non compact operator $T\colon X \to Y$ that factorizes through a Banach space Z that is a closed subspace of a Banach space with an (uncountable or countable) (u.c.e.i)?

Question 4.8. Is it true that c_0 embeds into $K(X,Y)$ only if there is a non compact operator $T\colon X \to Y$ that factorizes through a Banach space Z that is a closed subspace of a Banach space with an (uncountable or countable) (u.c.e.i)?

Question 4.9. Which are the most general assumptions on X and Y under which conditions a)–e) (see Introduction) are equivalent?

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Authors' addresses: G. Emmanuele, Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy; K. John, Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic.