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On complemented copies of c_0 in spaces of operators

Abstract. We present a general procedure for getting complemented copies of c_0 in spaces of operators between two spaces E and F, assuming one of them or both containing a (not necessarily complemented) copy of c_0, too. This procedure makes use of a characterization of complemented copies of c_0 in general Banach spaces obtained by Schlumprecht ([4]). Some results due to E. and P. Saab and to R. Ryan are corollaries of our main theorem.

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Let E and F be two Banach spaces. By $L(E^*, F)$ we denote the B-space of all operators from E^* into F. Purpose of this note is to present a general procedure for getting complemented copies of c_0 inside of suitable closed subspaces of $L(E^*, F)$ (containing $E \otimes_\varepsilon F$), once one of E and F^* or both contain a (not necessarily complemented) copy of c_0. Some results of this kind have been already obtained by Cembranos ([1]), E. and P. Saab ([3]) and R. Ryan ([2]); their theorems concern with spaces of compact operators. Here we obtain a general theorem from which these results easily follow as well as other ones concerning special spaces of operators. Our main result makes use of a characterization of complemented copies of c_0 in general B-spaces due to Schlumprecht ([4]).

The present results were obtained during the stay of the author at Kent State University in 1988, thanks to a C.N.R.-fellowship. We take this opportunity to thank R. Aron and J. Diestel for helpful discussions about the subject.

1. Before starting with the proof of the main theorem, we have to recall some definition and the quoted theorem of Schlumprecht

DEFINITION. A (bounded) subset X of a B-space E is called *limited* if, for each w^*-null sequence $(x_n^*) \subset E^*$, one has

$$\lim_n \sup_X |x_n^*(x)| = 0.$$

THEOREM 1 ([4]). *If a B-space E contains a copy (x_n) of the unit vector basis of c_0 that is not limited, then E contains a complemented copy of c_0 spanned by a subsequence of (x_n).*

We need some notation. Let $\mathcal{F}$ be a family of subsets of a B-space F such that

(j) $\mathcal{F}$ contains the set of the form $\{\alpha y : \alpha \in J\}$ where J is an arbitrary bounded real interval and y is arbitrary in F

(jj) for each $\alpha_1, \alpha_2 \in \mathbb{R}$, $A_1, A_2 \in \mathcal{F}$ the set

$$\alpha_1 A_1 + \alpha_2 A_2 = \{\alpha_1 y_1 + \alpha_2 y_2 : y_1 \in A_1, \ y_2 \in A_2\}$$

again belongs to $\mathcal{F}$.

We note that (j) means exactly that S defined below in Theorem 2 contains all one dimensional operators and (jj) means that the same S is a linear space.

Moreover, assume that given $\mathcal{F}$ as above, there is a bounded sequence (y_n^*) in F^* such that

(jjj) $\inf_n \|y_n^*\| > 0$

(jv) $\lim_n \sup_A |y_n^*(y)| = 0$ for $A \in \mathcal{F}$.

THEOREM 2. *Let E, F be two B-spaces and $(\mathcal{F}, (y_n^*))$ be as above. Assume $(x_n) \subset E$ is a copy of the unit vector basis of c_0 and, further, the following subspace of $L(E^*, F)$ is closed*

$$S = \{T : T \in L(E^*, F), T(B_{E^*}) \in \mathcal{F}\}.$$

Then S contains a complemented copy of c_0.

Proof. Consider a bounded sequence $(x_n^*) \subset E^*$ of coefficient functionals associated to (x_n) and a sequence $(y_n) \subset B_F$ so that $\inf_n |y_n^*(y_n)| > 0$. The sequence $(x_n \otimes y_n)$ is a copy of the unit vector basis of c_0 (see [2]) and it belongs to S because of (j). Moreover $(x_n^* \otimes y_n^*) \subset S^*$ is bounded and $\inf_n |(x_n^* \otimes y_n^*)(x_n \otimes y_n)| > 0$. If we prove that $x_n^* \otimes y_n^* \xrightarrow{w^*} \theta$ an appeal to Theorem 1 will conclude the proof. We have, for $T \in S$ and $n \in \mathbb{N}$, $n \to \infty$,

$$|T(x_n^* \otimes y_n^*)| = |T(x_n^*)(y_n^*)| \leqslant |T(x_n^*/\|x_n^*\|)(y_n^*)| \sup_n \|x_n^*\| \to 0$$

since $T(x_n^*/\|x_n^*\|) \in T(B_{E^*})$ and (jv) holds. We are done.

2. We now present several corollaries following from the main Theorem 2 concerning different subspaces S we can apply our result to (for suitable choices of the pair $(\mathcal{F}, (y_n^*))$).

$\mathcal{F}$ = {relatively compact subsets of F} and (y_n^*) a weak* null, norm one sequence.

In this case we get the results in [2] and [3]. We also have the following

COROLLARY 1. *Let X, F (infinite dimensional) B-spaces. If X^* contains a copy of c_0, then $K(X,F) = \{compact\ operators\}$ contains a complemented copy of c_0.*

The Referee of this paper suggested the following result following from Corollary 1; we take this opportunity to thank him very much for this Corollary 2

COROLLARY 2. *Let E^*, F^* contain a copy of c_0 and let $B(E,F)$ be an arbitrary closed space of operators such that*

$$\mathcal{F}(E,F) \subseteq B(E,F) \subseteq \mathcal{R}(E,F)$$

(here $\mathcal{F}(E,F)$ is the space of all finite dimensional operators and $\mathcal{R}(E,F)$ that one of all Rosenthal operators, i.e. operators which do not fix any copy of l_1). Then $B(E,F)$ contains a complemented copy of c_0.

Proof. If F^* contains c_0, then F contain a complemented copy G of l_1, where $P : F \to G$ is a projection onto. Moreover, since $K(E,l_1) = \mathcal{R}(E,l_1)$ we have $\{T : T \in B(E,F), T(E) \subseteq G\} = K(E,G)$. Finally, the map $H : B(E,F) \to K(E,G)$, $H(T) = P \circ T$ is a projection onto and we apply Corollary 1 to $K(E,G)$. We are done.

It is worth noting that above corollary holds for B = the class of all weakly compact operators.

The method of the proof of Theorem 2 has an interesting consequence concerning the completion of the space of Pettis integrable functions. Let (S,Σ,μ) be a finite measure space and $P(\mu,E)$ the space of all Pettis integrable functions from S into E equipped with the norm

$$((f)) = \sup_{B_{E^*}} \int_S |x^* f(s)|\, d\mu.$$

COROLLARY 3. *If E contains a copy of c_0, then the completion of $P(\mu,E)$ contains a complemented copy of c_0.*

Proof. We prove that $P(\mu,E)$ (and hence its completion) is a (closed) subspace of $W(E^*, L^1(\mu))$. For each $f \in P(\mu,E)$ we consider the operator $T_f : E^* \to L^1(\mu)$ defined by $T_f(x^*) = x^* f$. Using the fact that the indefinite integral of f has a relatively weakly compact range, it is not difficult to see that T_f is weakly compact. On the other hand it is very easy to prove that the mapping assigning to each $f \in P(\mu,E)$ the operator T_f is an isometry (we identify functions that are weakly equivalent). Now, observe that if $(x_n) \subset E$ is a copy of the unit vector basis of c_0 and (y_n^*) is another copy of the unit vector basis of c_0 in $L^\infty(\mu) = (L^1(\mu))^*$ we can choose a bounded sequence (y_n) in $L^1(\mu)$ with $\inf |y_n^*(y_n)| > 0$. Hence the proof of Theorem 2 works.

Corollary 3 improves an (unpublished) result obtained by J. Diestel in 1988 showing that if E contains c_0, then the completion of the space of Pettis integrable functions having indefinite integrals with a relatively compact range contains a complemented copy of c_0.

At the end we present some more consequence of Theorem 2, without presenting their (standard) proof.

COROLLARY 4. *Let E contain a copy of c_0. If F^* is not a Schur space, then the closed subspace S of all $T \in L(E^*, F)$, such that T^* is a Dunford-Pettis operator, contains a complemented copy of c_0.*

COROLLARY 5. *Let E and F^* contain a copy of c_0. Then the closed subspace S of all $T \in L(E^*, F)$, such that T^* is an unconditionally converging operator, contains a complemented copies of c_0.*

COROLLARY 6. *Let F be the dual space of a B-space Z. If E contains a copy of c_0 and Z is not a Schur space, then the closed subspace S of all $T \in L(E^*, F)$, such that the restriction of T^* to Z is a Dunford-Pettis operator, contains a complemented copy of c_0.*

COROLLARY 7. *Let F be the dual space of a B-space Z. If E and Z contain a copy of c_0, then the closed subspace S of all $T \in L(E^*, F)$, such that the restriction of T^* to Z is an unconditionally converging operator, contains a complemented copy of c_0.*

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