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COMPACTNESS OF DOMINATED OPERATORS AND THE (CRP) IN SPACES OF COMPACT OPERATORS

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ABSTRACT. We present results showing that sometimes the (CRP) lifts from two Banach spaces E^*, F to the Banach space $K(E,F)$. They essentially depend on the compactness of certain dominated operators.

KEYWORDS: *Spaces of compact operators, compact range property.*

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Let E, F be two Banach spaces. Several papers have been devoted to the question of when an isomorphic property lifts from the spaces E^*, F to the space of compact operators $K(E,F)$ (see [1], [5], [8], [12], [20] and References); following the same line of research we devote this paper to the question of when the so called *Compact Range Property*, in symbols (CRP), is enjoyed by $K(E,F)$. From our results it follows that this question is heavily depending on the question of when each *dominated operator* from the space $C(S,E)$, S a Hausdorff compact space, to F is compact. In the case $E = \mathbb{R}$ the results in [11] show that these two questions actually are equivalent, but in the infinite dimensional case we do not know if the same happens or not, except in the case of F a dual Banach space in which we still have equivalence.

In order to start we need three definitions

DEFINITION 1. ([17]) A Banach space E is said *to possess the (CRP)* if any E-valued countably additive measure μ, defined on a σ-field of subsets of an arbitrary set S, with finite variation $|\mu|(S)$, has relatively compact range.

DEFINITION 2. ([7]) An operator $T : C(S, E) \to F$, S a Hausdorff compact space, is called a *dominated operator* if there exists a countably additive regular positive Borel measure μ such that

$$\|T(f)\|_F \leq \int_S \|f(t)\|_E \mathrm{d}\mu \qquad f \in C(S, E).$$

DEFINITION 3. Let E be a Banach space. A bounded subset X of E is a *Dunford-Pettis set* if $\lim_n \sup_X |x_n^*(x)| = 0$ for every w-null sequence (x_n^*) in E^*.

The first result, which is also the main result of the paper, clarifies the role of dominated operators in the investigation of the (CRP) in $K(E, F)$. In it (and in the sequel too) we shall make use of the following well known equivalence: *E does not contain copies of l_1 if and only if E^* has the* (CRP) (see, for instance, [11]).

THEOREM 1. *Let E^* have the* (CRP). *If, for any Hausdorff compact space S, each dominated operator $T : C(S, E) \to F$ is compact, we get*

(i) $L(E, F) = K(E, F)$

(ii) *$K(E, F)$ has the (CRP).*

Proof. (i) If H is not a compact operator from E into F, then for an arbitrary S and an arbitrary μ (as in the definition above) we can define a dominated operator $T : C(S, E) \to F$ by putting

$$T(f) = \int_S Hf(t)\mathrm{d}\mu \qquad f \in C(S, E).$$

Using constant functions we see very easily that T is not compact.

(ii) We still argue by contradiction. Let S be a set, Σ a σ-algebra of subsets of S and μ a countably additive measure with bounded variation from Σ into the space $K(E, F)$ without relatively compact range. Using the Stone representation Theorem ([7]) we can assume that S is a Hausdorff compact space and μ is a regular Borel measure on Σ. Since μ does not have relatively compact range there is a sequence (A_n) in $\mathrm{Bo}(S)$ such that $(\mu(A_n))$ converges weakly to some $H \in K(E, F)$ (because the range of any vector measure is relatively weakly compact, [6]), but not strongly. So there is a sequence $(x_n) \subset B_E$ such that $(\mu(A_n)(x_n) - H(x_n))$ does not go to zero strongly in F. Since E does not contain l_1 we can assume that (x_n) is a weak Cauchy sequence in B_E, otherwise we pass to a subsequence (use the famous Rosenthal Theorem, [19]). Let $x^{**} \in E^{**}$ be the w*-limit of (x_n) in E^{**}. We want to show that $(\mu(A_n)(x_n) - H(x_n))$ goes to θ weakly in F. Take $y^* \in F^*$. We consider the following equality

$$(1)\qquad \begin{aligned}\langle \mu(A_n)(x_n) - H(x_n), y^* \rangle &= \langle [\mu(A_n)]^*(y^*), x_n - x^{**} \rangle - \langle H^*(y^*), x_n - x^{**} \rangle \\ &\quad + \langle \mu(A_n) - H, x^{**} \otimes y^* \rangle \qquad n \in \mathbb{N}.\end{aligned}$$

Since $\mu(A_n) \xrightarrow{w} H$ and $x^{**} \otimes y^* \in (K(E,F))^*$ the last summand of (1) goes to zero as well as the second one does because $x_n \xrightarrow{w^*} x^{**}$ in E^{**} and $H^*(y^*) \in E^*$. Now, we recall that the range of a vector measure μ of bounded variation is a Dunford-Pettis set ([12]). Using the operator $P : K(E,F) \to E^*$ defined by $P(H) = H^*(y^*)$ we see very easily that $([\mu(A_n)]^*(y^*))$ is a Dunford-Pettis set in E^*. From [11] and [12] it follows that E^* has the (CRP) if and only if any Dunford-Pettis set in E^* is relatively compact; since $x_n \xrightarrow{w^*} x^{**}$ we see very easily that also the first summand of (1) goes to zero. This shows that $(\mu(A_n)(x_n) - H(x_n))$ goes to θ weakly in F. Since H is compact, we can assume that there is $y \in F$ such that $\|H(x_n) - y\| \to 0$. Hence $\mu(A_n)(x_n) \xrightarrow{w} y$, but not strongly. Now, define a dominated operator $T : C(S,E) \to F$ by

$$T(f) = \int_S f(t)\, d\mu \qquad f \in C(S,E).$$

T can be extended to all of $L^1(|\mu|, E)$ because it is dominated and μ is regular. If we consider the functions $g_n = \chi_{A_n} x_n$ in $L^1(|\mu|, E)$, using Lusin Theorem and Borsuk–Dugundji Extension Theorem we can approximate each of them by elements in the unit ball of $C(S,E)$ in the L^1-norm. But T must be compact and so the above remarks imply that $(T(g_n)) = (\mu(A_n)(x_n))$ must be relatively compact in F. This contradicts what we proved. We are done. ∎

QUESTION 1. Is the converse of Theorem 1 true?

Theorem 1 can be used to get the following sufficient conditions for $K(E,F)$ to possess the (CRP). For the first of these results we need one more definition

DEFINITION 4. ([9]) Let E be a Banach space. A bounded subset X of E^* is called a *(L) set* if $\lim_n \sup_X |x_n(x^*)| = 0$ for every w-null sequence (x_n) in E.

THEOREM 2. *Let F be a dual Banach space. If E^*, F have the* (CRP) *and $L(E,F) = K(E,F)$, then $K(E,F)$ has the* (CRP).

Proof. We shall prove that under our assumptions any dominated operator T from $C(S,E)$ into F is compact. We need a representation theorem for dominated operators to be found in [7]. According to it there is $G : S \to L(E,F) = K(E,F)$ such that

(i) $|G(t)| = 1$ almost everywhere on S

(ii) for each $y \in Z$, $Z^* = F$, and $f \in C(S,E)$ the function $\langle G(\cdot)f(\cdot), y\rangle$ is μ-integrable and furthermore

$$\langle T(f), y\rangle = \int_S \langle G(t)f(t), y\rangle \, \mathrm{d}\mu \qquad \forall f \in C(S,E),\ y \in Z \tag{2}$$

where μ is the least positive regular Borel measure dominating T.

We shall prove that any sequence in $T(B_{C(S,E)})$ is an (L)-set. So let us consider $(f_n) \subset B_{C(S,E)}$ and $(y_n) \subset B_Z$ such that $y_n \xrightarrow{\mathrm{w}} \theta$. It is clear that $G^*(t)y_n \xrightarrow{\mathrm{s}} \theta$ on S and so we have $\lim_n \langle G(t)f_n(t), y_n\rangle = 0$. Since $|\langle G(t)f_n(t), y_n\rangle| \leqslant 1$ a. e. on S, for all $n \in \mathbb{N}$, we get

$$\lim_n \int_S \langle G(t)f_n(t), y_n\rangle \, \mathrm{d}\mu = 0. \tag{3}$$

Both (2) and (3) imply that $T(B_{C(S,E)})$ is an (L)-set in F, i. e. a relatively compact subset of F, by virtue of a result in [9]. We are done, thanks to Theorem 1. ∎

REMARK. Theorem 2 was also obtained in [12] with a totally different proof.

In the case considered above of F a dual Banach space we are also able to answer positively Question 1 thanks to the following result in which we use an isomorphic property introduced in our paper [12].

DEFINITION 5. ([12]) A Banach space E is said to *have the (DPrcP)* if any Dunford-Pettis set in E is relatively compact.

THEOREM 3. *Let F be a dual Banach space such that $L(E,F) = K(E,F)$ and $K(E,F)$ has the* (CRP). *Then any dominated operator T from $C(S,E)$ into F is compact.*

Proof. From [12] it follows that the (CRP) and the (DPrcP) are equivalent in dual spaces; so, from our hypotheses, it follows that F has the (DPrcP). To reach our goal it will be enough to prove that a dominated operator $T : C(S,E) \longrightarrow F$ maps $B_{C(S,E)}$ into a Dunford-Pettis subset. This can be done as in Theorem 2. We are done. ∎

Since in the case of F not a dual space the function G takes its values in $L(E, F^{**})$, we have the following result in which we still use Definition 5.

THEOREM 4. *Let E^* have the* (CRP) *and F the* (DPrcP). *If $L(E,F) = K(E,F)$ then $K(E,F)$ has the* (CRP).

Proof. The proof goes as in Theorem 2 with the only change that $T(B_{C(S,E)})$ is now a Dunford-Pettis set. ∎

In the next result we use the definition of Gelfand-Phillips property (see, for instance, [8]).

DEFINITION 6. A Banach space E is said to have the *Gelfand-Phillips property*, in symbols (GPP), if any limited set in E is relatively compact. A bounded subset X of E is a *limited set* if $\limsup_n \sup_X |x_n^*(x)| = 0$ for every w*-null sequence (x_n^*) in E^*.

We shall also use the well known definition of Radon-Nikodym property (in symbols (RNP)) for which we refer to [6], where one can also find the well known fact that the (RNP) implies the (CRP).

THEOREM 5. *Let E be a separable complemented Banach space with E^* enjoying the* (CRP) *and F be a Banach space possessing the* (RNP). *If $L(E,F) = K(E,F)$, then $K(E,F)$ has the* (CRP).

Proof. Let T be a dominated operator from $C(S,E)$ into F. If (f_n) is a sequence in $B_{C(S,E)}$ we consider $E_0 = \overline{\mathrm{span}}\{f_n(t) : n \in \mathbb{N},\ t \in S\}$; note that it is a separable Banach space and we can assume that E_0 is (contained in) a complemented and separable subspace of E; hence we have the equality $L(E_0, F) = K(E_0, F)$. Furthermore, following [2], Theorem 8, we can suppose that S is a compact metric space. Hence, $C(S, E_0)$ is separable, which implies that T actually takes its values in a separable subspace of F; our assumptions allow us to suppose that F is separable (and hence that it has the (GPP),[8]) and it enjoys the (RNP). If G is the representing function of T and ρ is a lifting of $L_\infty(\mu)$, μ the least countably additive regular Borel measure dominating T, we can choose G so that

$$\rho(\langle G(\cdot)x, y^*\rangle) = \langle G(\cdot)x, y^*\rangle \qquad \forall x \in E,\ y^* \in F^*.$$

Now, let us consider a countable subset (z_n) dense in E_0; if we denote by $m : \mathrm{Bo}(S) \longrightarrow K(E_0, F)$ the representing measure of T, the measures $m(\cdot)z_n$ take their values in F, that has the (RNP). Theorem III. 2.7 in [6] gives that, for each $\varepsilon > 0$ there exists a set S_ε, with $\mu(S \setminus S_\varepsilon) < \varepsilon$ and such that

$$C_\varepsilon = \overline{\mathrm{co}}\left\{ \frac{m(H)z_n}{\mu(H)} : H \in \mathrm{Bo}(S), H \subset S_\varepsilon, \mu(H) > 0 \right\}$$

is a compact subset of F, for each $n \in \mathbb{N}$. Repeating the proof of part (4,b) in [7], Chapter II, Section 13.4, Theorem 4, we get the existence of a null subset S_0 of S for which $G(t)z \in F$ for all $t \in S \setminus S_0$ and all $z \in E_0$ (use also the separability of E_0). Now, considering a w*-null sequence (y_n^*) in F^*, as in Theorem 2 we show that $T(B_{C(S,E_0)})$ is a limited subset of F and so relatively compact. We are done. ∎

Theorem 5 generates the following natural double question:

QUESTION 2. Is it possible to eliminate the assumption "E is separably complemented"? Is it possible to improve the assumption on F just assuming that it has the (WRNP) (see [17]) ?

We can partially answer to the first of these questions using different assumptions on E or F.

THEOREM 6. *Let E be a Banach space with E^* enjoying the (CRP) and F be a Banach space possessing the (RNP). Let us also suppose that at least one of the following conditions is verified:*

(i) *E has property* (u) *due to Pełczynski (see* [18]*),*

(ii) *F is weakly sequentially complete,*

(iii) *E^* has Schur property and in F each Dunford-Pettis set is relatively weakly compact.*

If $L(E,F) = K(E,F)$, *then* $K(E,F)$ *has the* (CRP).

Proof. As in Theorem 5 we may reduce ourselves to the case of a separable subspace E_0 of E, of a compact metric space S and a separable F. Proposition 3.4 in [15] allows us to suppose that there is a isometric embedding J of E_0^* into E^*. Looking at the proof of Theorem 5 we easily realize that it still works once we have shown that $K(E_0,F) = L(E_0,F)$. So let us consider an element $T \in L(E_0,F)$. Under any of the assumptions (i) and (ii) it is easily seen that T is weakly compact. Indeed, if i) is true we get that E_0 has property (V) of Pełczynski ([18]) as any space not containing copies of l_1 with property (u) does; now we are done since F does not contain copies of c_0. Whereas if (ii) is true we are still done since E_0 does not contain copies of l_1 and F is weakly sequentially complete. Hence, the operator T^* is weakly compact and so a weak*-weak continuous operator from F^* into E_0^*, that composed with J still gives a weak*-weak continuous operator from F^* into E^*, i. e. a conjugate operator. But each such an operator must be compact because of our assumption $L(E,F) = K(E,F)$; since J is an isometry we can conclude that T^*, and so T, must be compact. In the case (iii) is true, since J is an isometry from E_0^* into E^* we argue that even E_0^* has Schur property. It is thus very easy to see that $T(B_{E_0})$ is a Dunford-Pettis subset of F and hence a relatively weakly compact set. Hence T is weakly compact and we are done thanks to the previous reasonings. ∎

In all the results above we assumed the most general hypothesis allowed on E, i. e. the (CRP) for E^*, but something less general about F; in the next result we try to reverse this situation, which means that we shall assume that F has the (CRP); but this will force us to choose a very particular E. Nevertheless the next Theorem 7 will have an interesting consequence that will be stated at the end of the paper as Corollary 8.

THEOREM 7. *Let* F *have the* (CRP). *Then* $K(c_0,F)$ *has the* (CRP).

Proof. From Theorem 1 it is enough to show that any dominated operator T from $C(S,c_0)$ to F is compact. Take $f \in C(S,c_0)$. For each $t \in S$ there is $f_n(t) \in \mathbb{R}$ such that $f(t) = \sum f_n(t)e_n$, (e_n) the unit vector basis of c_0. It is clear that each f_n is continuous on S and that (f_n) is equicontinuous and equibounded in $C(S)$; since (f_n) goes to zero pointwise, it converges in the sup-norm to θ. So any $f \in C(S,c_0)$ can be identified with an element of $c_0(C(S))$ and, actually, this identification is an isometry onto. Now, observe that c_0 is not contained in F and so any operator $T : C(S,c_0) \to F$ is weakly compact since $C(S,c_0)$ has property

(V) of Pełczynski (see [18]). Defining $T_n : C(S) \to F$ by putting $T_n(f) = T((f_h))$, where $f_h = f$ if $h = n, 0$ otherwise, we get that T_n is weakly compact and

$$\lim_m \left\| \sum_{i=1}^{m} T_i \circ P_i - T \right\| = 0 \tag{4}$$

thanks to a result in [3] (in (4) P_i denotes the i-th projection of $C(S, c_0) = c_0(C(S))$ onto its i-th factor). Furthermore, each T_n is clearly dominated and hence compact, since F has the (CRP) ([11]). It follows from (4) above that T is compact. We are done. ∎

It is clear that in Theorem 7 we still have $L(c_0, F) = K(c_0, F)$.

In Theorems 2–7 we used the assumption $L(E, F) = K(E, F)$ in order to guarantee that any dominated operator from $C(S, E)$ into F is compact and, consequently, that $K(E, F)$ has the (CRP), thanks to Theorem 1. This assumption is, in several many cases, necessary for $K(E, F)$ to possess the (CRP) as may be deduced from a number of results from the papers [10], [14], [16], because if it does not hold then c_0 embeds into $K(E, F)$ that it is not allowed to possess the (CRP). However we must underline that, as remarked in the paper [13], at least in one case $K(E, F)$ has the (CRP), actually the (RNP), even if $L(E, F) \neq K(E, F)$; it is enough to take $E = F =$ a $\mathcal{L}_\infty$-space with the (RNP) and with dual isomorphic to l_1 constructed in [4] by Bourgain and Delbaen. We also remark that the same Bourgain-Delbaen space furnishes an example of pair E, F such that $K(E, F)$ has the (CRP), $K(E, F) \neq L(E, F)$ and there exists a not compact dominated operator from $C(S, E)$ to F, so that it cannot be used to answer Question 1. Following the lines for the proof of this fact, contained in [13], we can get the last result of the paper that also uses Theorem 7 as already announced.

COROLLARY 8. *Let E be a Banach space such that E^* is isomorphic to l_1. If F has the* (CRP), *then $K(E, F)$ has the same property.*

Proof. $K(E, F)$ is isomorphic to $E^* \otimes_\varepsilon F$, that in turn is isomorphic to $l_1 \otimes_\varepsilon F$, that in turn is isomorphic to $K(c_0, F)$. It is now enough to apply Theorem 7. ∎

Once more, we remark that the assumptions of Corollary 8 do not necessarily imply that $L(E, F) = K(E, F)$.

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