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THE PROJECTIVE AND INJECTIVE TENSOR PRODUCTS OF $L^p[0,1]$ AND X BEING GROTHENDIECK SPACES

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ABSTRACT. Let X be a Banach space and $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$. Then $L^p[0,1]\hat{\otimes}X$, respectively $L^p[0,1]\check{\otimes}X$, the projective, respectively injective, tensor product of $L^p[0,1]$ and X, is a Grothendieck space if and only if X is a Grothendieck space and each continuous linear operator from $L^p[0,1]$, respectively $L^{p'}[0,1]$, to X^*, respectively X^{**}, is compact.

1. Introduction. In [**1, 4, 5**], Bu, Diestel, and Dowling gave a sequential representation of $L^p[0,1]\hat{\otimes}X$, the projective tensor product of $L^p[0,1]$ and X when $1 < p < \infty$. By this sequential representation, they showed that $L^p[0,1]\hat{\otimes}X$, $1 < p < \infty$, has the Radon-Nikodym property (respectively the analytic Radon-Nikodym property, the near Radon-Nikodym property, contains no copy of c_0) if and only if X has the same property. Using this sequential representation, Bu in [**2**] showed that $L^p[0,1]\hat{\otimes}X$, $1 < p < \infty$, contains no copy of l_1 if and only if X contains no copy of l_1 and each continuous linear operator from $L^p[0,1]$ to X^* is compact, and he also in [**3**] discussed all these geometric properties in $L^p[0,1]\check{\otimes}X$, the injective tensor product of $L^p[0,1]$ and X when $1 < p < \infty$.

In [**9**], Emmanuele showed that if X and Y are Grothendieck Banach spaces, one of which is reflexive, and if each continuous linear operator from X to Y^* is compact, then $X\hat{\otimes}Y$, the projective tensor product of X and Y, is a Grothendieck space. And he also in [**10**] showed that if $X\hat{\otimes}Y$ is a Grothendieck space and Y^* has the (b.c.a.p), then each continuous linear operator from X to Y^* is compact. As a special case of Emmanuele's results, we have that if X has the (b.c.a.p), then $L^p[0,1]\hat{\otimes}X$, $1 < p < \infty$, is a Grothendieck space if and only if X is a Grothendieck space and each continuous linear operator from $L^p[0,1]$

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to X^* is compact. In this paper, through the sequential representation of $L^p[0,1]\hat{\otimes}X$, we give a new proof of Emmanuele's special case and, meanwhile, we characterize $L^p[0,1]\hat{\otimes}X$ and $L^p[0,1]\check{\otimes}X$, $1 < p < \infty$, being Grothendieck spaces for any Banach space X.

2. Preliminaries. For $1 < p < \infty$, let p' denote its conjugate, i.e., $1/p + 1/p' = 1$. For a sequence $\bar{x} = (x_i)_i \in X^{\mathbf{N}}$ and $n \in \mathbf{N}$, denote

$$\bar{x}(i > n) = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots).$$

For any Banach space X, we will denote its topological dual by X^* and its closed unit ball by B_X. For two Banach spaces X and Y, let $\mathcal{L}(X,Y)$ denote the space of all continuous linear operators from X to Y, $\mathcal{K}(X,Y)$ the space of all compact operators from X to Y, and $\mathcal{N}(X,Y)$ the space of all nuclear operators from X to Y.

From [**12**, p. 3] and [**13**, p. 155], we know that the Haar system $\{\chi_i\}_{i=1}^\infty$ is an unconditional basis of $L^p[0,1]$ for $1 < p < \infty$. Let us use K_p to denote the unconditional basis constant of the basis $\{\chi_i\}_{i=1}^\infty$. Now renorm $L^p[0,1]$ by

$$\|f\|_p^{\mathrm{new}} = \sup\left\{\left\|\sum_{i=1}^\infty \theta_i a_i \chi_i\right\|_p : \theta_i = \pm 1,\ i = 1, 2, \dots\right\},$$
$$f = \sum_{i=1}^\infty a_i \chi_i \in L^p[0,1].$$

Then

$$\|\cdot\|_p \le \|\cdot\|_p^{\mathrm{new}} \le K_p \cdot \|\cdot\|_p.$$

With this new norm, $L^p[0,1]$ is also a Banach space. Furthermore, $\{\chi_i\}_{i=1}^\infty$ is a monotone, unconditional basis with respect to this new norm. Now let

$$e_i = \frac{\chi_i}{\|\chi_i\|_p^{\mathrm{new}}}, \quad i = 1, 2, \dots.$$

Then $\{e_i\}_{i=1}^\infty$ is a normalized, unconditional basis of $(L^p[0,1], \|\cdot\|_p^{\mathrm{new}})$ whose unconditional basis constant is 1. For convenience, let

$$e_i^* = \frac{\chi_i}{\|\chi_i\|_{p'}^{\mathrm{new}}}, \quad i = 1, 2, \dots.$$

From [**12**, pp. 18–19] we have the following

Proposition 1. *Let* $u = \sum_{i=1}^{\infty} e_i^*(u) e_i \in L^p[0,1]$, $1 < p < \infty$. *Then*

(i) *For each subset* σ *of* $\mathbf{N}$, $\| \sum_{i\in\sigma} e_i^*(u) e_i \|_p^{\mathrm{new}} \le \|u\|_p^{\mathrm{new}}$.

(ii) *For each choice of signs* $\theta = \{\theta_i\}_1^{\infty}$, $\|\sum_{i=1}^{\infty} \theta_i e_i^*(u) e_i\|_p^{\mathrm{new}} \le \|u\|_p^{\mathrm{new}}$.

(iii) *For each* $\lambda = (\lambda_i)_i \in l_\infty$, $\|\sum_{i=1}^{\infty} \lambda_i e_i^*(u) e_i\|_p^{\mathrm{new}} \le 2 \cdot \|\lambda\|_{l_\infty} \cdot \|u\|_p^{\mathrm{new}}$.

For any Banach space X and $1 < p < \infty$ with $1/p + 1/p' = 1$, define

$$L^p_{\mathrm{weak}}(X) = \Big\{ \bar{x} = (x_i)_i \in X^{\mathbf{N}} : \sum_i x^*(x_i) e_i \text{ converges in } L^p[0,1] \ \forall\, x^* \in X^* \Big\},$$

$$L^p\langle X\rangle = \Big\{ \bar{x} = (x_i)_i \in X^{\mathbf{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty \ \forall\, (x_i^*)_i \in L^{p'}_{\mathrm{weak}}(X^*) \Big\};$$

and define norms on $L^p_{\mathrm{weak}}(X)$ and $L^p\langle X\rangle$, respectively, to be

$$\|\bar{x}\|_{L^p_{\mathrm{weak}}(X)} = \sup \left\{ \left\| \sum_{i=1}^{\infty} x^*(x_i) e_i \right\|_p^{\mathrm{new}} : x^* \in B_{X^*} \right\}, \quad \bar{x} \in L^p_{\mathrm{weak}}(X),$$

$$\|\bar{x}\|_{L^p\langle X\rangle} = \sup \left\{ \sum_{i=1}^{\infty} |x_i^*(x_i)| : (x_i^*)_i \in B_{L^{p'}_{\mathrm{weak}}(X^*)} \right\}, \quad \bar{x} \in L^p\langle X\rangle.$$

With their own norm, respectively, $L^p_{\mathrm{weak}}(X)$ and $L^p\langle X\rangle$ are Banach spaces [**1, 4**]. Let $L^p_{\mathrm{weak},0}(X)$ denote the closed subspace of $L^p_{\mathrm{weak}}(X)$ such that the tail of each member of $L^p_{\mathrm{weak},0}(X)$ converges to zero, i.e.,

$$L^p_{\mathrm{weak},0}(X) = \Big\{ \bar{x} = (x_i)_i \in L^p_{\mathrm{weak}}(X) : \lim_n \|\bar{x}(i > n)\|_{L^p_{\mathrm{weak}}(X)} = 0 \Big\}.$$

From [**1**] we have the following proposition.

Proposition 2. (i) *For each* $\bar{x} = (x_i)_i \in L^p\langle X\rangle$,

$$\lim_n \|\bar{x}(i > n)\|_{L^p\langle X\rangle} = 0.$$

(ii) $L^p[0,1]\hat{\otimes}X$ *is isomorphic to* $(L^p[0,1], \|\cdot\|_p^{\text{new}})\hat{\otimes}X$ *which is isometrically isomorphic to* $L^p\langle X\rangle$.

Proposition 3. $L^p_{\text{weak}}(X)$ *is isometrically isomorphic to*

$$\mathcal{L}((L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}}), X).$$

Proof. Define

$$\begin{aligned} \phi : L^p_{\text{weak}}(X) &\longrightarrow \mathcal{L}((L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}}), X) \\ \bar{x} &\longmapsto \phi(\bar{x}), \end{aligned}$$

where, for each $\bar{x} = (x_i)_i \in L^p_{\text{weak}}(X)$,

$$\begin{aligned} \phi(\bar{x}) : (L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}}) &\longrightarrow X \\ u^* &\longmapsto \sum_{i=1}^{\infty} u^*(e_i)x_i\,. \end{aligned}$$

Let $u^* \in (L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}})$ and $n, m \in \mathbf{N}$ with $m > n$. Then

$$\begin{aligned} \Big\|\sum_{i=n}^{m} u^*(e_i)x_i\Big\|_X &= \sup\Big\{\Big|\sum_{i=n}^{m} u^*(e_i)x^*(x_i)\Big| : x^* \in B_{X^*}\Big\} \\ &= \sup\Big\{\Big|\Big\langle \sum_{i=n}^{m} u^*(e_i)e_i^*, \sum_{i=1}^{\infty} x^*(x_i)e_i\Big\rangle\Big| : x^* \in B_{X^*}\Big\} \\ &\le \sup\Big\{\Big\|\sum_{i=n}^{m} u^*(e_i)e_i^*\Big\|_{p'}^{\text{new}} \cdot \Big\|\sum_{i=1}^{\infty} x^*(x_i)e_i\Big\|_p^{\text{new}} : x^* \in B_{X^*}\Big\} \\ &= \|\bar{x}\|_{L^p_{\text{weak}}(X)} \cdot \Big\|\sum_{i=n}^{m} u^*(e_i)e_i^*\Big\|_{p'}^{\text{new}}. \end{aligned}$$

Since $\sum_i u^*(e_i)e_i^*$ converges in $(L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}})$, $\{\sum_{i=n}^{m} u^*(e_i)x_i\}_{n=1}^{\infty}$ is a Cauchy sequence in X and, hence, converges in X. So $\sum_{i=1}^{\infty} u^*(e_i)x_i \in X$ and

$$\Big\|\sum_{i=1}^{\infty} u^*(e_i)x_i\Big\|_X \le \|\bar{x}\|_{L^p_{\text{weak}}(X)} \cdot \|u^*\|.$$

Therefore ϕ is well defined and

$$\|\phi(\bar{x})\| \leq \|\bar{x}\|_{L^p_{\text{weak}}(X)}. \tag{1}$$

On the other hand, Let $T \in \mathcal{L}((L^{p'}[0,1], \|\cdot\|_{p'}^{\text{new}}), X)$. Define $x_i = T(e_i^*)$ for each $i \in \mathbf{N}$. By Proposition 1, for each $x^* \in X^*$ and each $n \in \mathbf{N}$,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} x^*(x_i)e_i\right\|_p^{\text{new}} \\
&\quad = \sup\left\{\left|\sum_{i=1}^{n} x^*(x_i)u^*(e_i)\right| : u^* \in B_{(L^{p'}[0,1],\|\cdot\|_{p'}^{\text{new}})}\right\} \\
&\quad \leq \|x^*\| \cdot \sup\left\{\left\|\sum_{i=1}^{n} u^*(e_i)x_i\right\|_X : u^* \in B_{(L^{p'}[0,1],\|\cdot\|_{p'}^{\text{new}})}\right\} \\
&\quad = \|x^*\| \cdot \sup\left\{\left\|\sum_{i=1}^{n} u^*(e_i)T(e_i^*)\right\|_X : u^* \in B_{(L^{p'}[0,1],\|\cdot\|_{p'}^{\text{new}})}\right\} \\
&\quad \leq \|x^*\| \cdot \|T\| \cdot \sup\left\{\left\|\sum_{i=1}^{n} u^*(e_i)e_i^*\right\|_{p'}^{\text{new}} : u^* \in B_{(L^{p'}[0,1],\|\cdot\|_{p'}^{\text{new}})}\right\} \\
&\quad \leq \|x^*\| \cdot \|T\| \cdot \sup\left\{\left\|\sum_{i=1}^{\infty} u^*(e_i)e_i^*\right\|_{p'}^{\text{new}} : u^* \in B_{(L^{p'}[0,1],\|\cdot\|_{p'}^{\text{new}})}\right\} \\
&\quad \leq \|x^*\| \cdot \|T\|.
\end{aligned}$$

So

$$\sup_n \left\|\sum_{i=1}^{n} x^*(x_i)e_i\right\|_p^{\text{new}} \leq \|x^*\| \cdot \|T\| < \infty. \tag{2}$$

Since $\{e_i\}_1^\infty$ is a boundedly complete basis of $L^p[0,1]$, the series $\sum_i x^*(x_i)e_i$ converges in $L^p[0,1]$ for each $x^* \in X^*$. Thus $\bar{x} = (x_i)_i \in L^p_{\text{weak}}(X)$. Moreover, $\phi(\bar{x}) = T$. Therefore ϕ is onto. Furthermore, from (2),

$$\|\bar{x}\|_{L^p_{\text{weak}}(X)} \leq \|T\| = \|\phi(\bar{x})\|. \tag{3}$$

Thus, combining (1) and (3), ϕ is an isometry. $\square$

Proposition 4. $L^p_{\text{weak},0}(X)$ *is isometrically isomorphic to*

$$\mathcal{K}((L^{p'}[0,1], \|\cdot\|^{\text{new}}_{p'}), X)$$

which is isomorphic to $L^p[0,1]\check{\otimes}X$.

Proof. For each $\bar{x} = (x_i)_i \in L^p_{\text{weak},0}(X)$, it is easy to see that its corresponding operator $T_{\bar{x}}$ is the limit of finite rank operators. So $T_{\bar{x}} \in \mathcal{K}((L^{p'}[0,1], \|\cdot\|^{\text{new}}_{p'}), X)$. On the other hand, if $T_{\bar{x}} \in \mathcal{K}((L^{p'}[0,1], \|\cdot\|^{\text{new}}_{p'}), X)$, then its adjoint operator $T^*_{\bar{x}} : X^* \to L^p[0,1]$ is compact. Note that, for each $x^* \in X^*$, $T^*_{\bar{x}}(x^*) = \sum_{i=1}^\infty x^*(x_i)e_i$. So $\{\sum_{i=1}^\infty x^*(x_i)e_i : x^* \in B_{X^*}\}$ is a relatively compact subset of $L^p[0,1]$. Thus,

$$\begin{aligned}\lim_n \|\bar{x}(i>n)\|_{L^p_{\text{weak}}(X)} &= \lim_n \sup\left\{\left\|\sum_{i=n+1}^\infty x^*(x_i)e_i\right\|^{\text{new}}_p : x^* \in B_{X^*}\right\}\\ &= 0.\end{aligned}$$

Hence $\bar{x} \in L^p_{\text{weak},0}(X)$. Therefore $L^p_{\text{weak},0}(X) = \mathcal{K}((L^{p'}[0,1], \|\cdot\|^{\text{new}}_{p'}), X)$. Note that $L^p[0,1]$ has the approximation property. Thus $\mathcal{K}(L^{p'}[0,1], X) = L^p[0,1]\check{\otimes}X$. □

It is known that, cf. [**8**, p. 230], $(L^p[0,1]\hat{\otimes}X)^*$ is isometrically isomorphic to $\mathcal{L}(L^p[0,1], X^*)$. Thus, from Proposition 2 and Proposition 3, we have

Proposition 5. $(L^p\langle X\rangle)^*$ *is isometrically isomorphic to* $L^{p'}_{\text{weak}}(X^*)$. *The dual operation is defined by*

$$\langle \bar{x}, \bar{x}^*\rangle = \sum_{i=1}^\infty x^*_i(x_i)$$

for each $\bar{x} = (x_i)_i \in L^p\langle X\rangle$ *and each* $\bar{x}^* = (x^*_i)_i \in L^{p'}_{\text{weak}}(X^*)$.

Note that $L^p[0,1]$ has the Radon-Nikodym property when $1 < p < \infty$. It is known from [**8**, pp. 232, 248, Theorem 6] that $(L^p[0,1]\check{\otimes}X)^*$ is isometrically isomorphic to $\mathcal{N}(L^p[0,1], X^*)$. Also note that $L^p[0,1]$ has the approximation property. It is also known from [**14**, p. 3] that $L^{p\prime}[0,1]\hat{\otimes}X$ is isometrically isomorphic to $\mathcal{N}(L^p[0,1], X)$. Thus, combining Proposition 2 and Proposition 4, we have

Proposition 6. $(L^p_{\mathrm{weak},0}(X))^*$ *is isometrically isomorphic to* $L^{p'}\langle X^*\rangle$. *The dual operation is defined by*

$$\langle \bar{x}, \bar{x}^* \rangle = \sum_{i=1}^{\infty} x_i^*(x_i)$$

for each $\bar{x} = (x_i)_i \in L^p_{\mathrm{weak},0}(X)$ *and each* $\bar{x}^* = (x_i^*)_i \in L^{p'}\langle X^*\rangle$.

3. Main results. Recall that a Banach space X is called a *Grothendieck space*, cf. [**6, 11**], if each separably valued bounded linear operator on X is weakly compact. By [**8**, p. 179] we know that a Banach space is a Grothendieck space if and only if any weak* convergent sequence in its dual space is weakly convergent.

Lemma 7. *Let* $\bar{x}^{(n)} = (x_i^{(n)})_i \in L^p_{\mathrm{weak},0}(X)$ *for each* $n \in \mathbf{N}$. *Then*

$$\sigma(L^p_{\mathrm{weak},0}(X), L^{p'}\langle X^*\rangle) - \lim_n \bar{x}^{(n)} = 0 \tag{4}$$

if and only if

$$\sigma(X, X^*) - \lim_n x_i^{(n)} = 0, \quad i = 1, 2, \dots \tag{5}$$

and

$$M = \sup_n \|\bar{x}^{(n)}\|_{L^p_{\mathrm{weak}}(X)} < \infty. \tag{6}$$

Proof. It is obvious that (4) $\Rightarrow$ (5) + (6). Next we want to show that (5) + (6) $\Rightarrow$ (4).

For each fixed $\bar{x}^* = (x_i^*)_i \in L^{p'}\langle X^*\rangle$ and each $\varepsilon > 0$, there exists, from Proposition 2, an $m \in \mathbf{N}$ such that

$$\|\bar{x}^*(i > m)\|_{L^{p'}\langle X^*\rangle} \le \varepsilon/2M.$$

From (5) there exists an $n_0 \in \mathbf{N}$ such that for each $n > n_0$,

$$|x_i^*(x_i^{(n)})| \le \varepsilon/2m, \quad i = 1, 2, \dots, m.$$

Thus, for each $n > n_0$,

$$\begin{aligned} |\langle \bar{x}^{(n)}, \bar{x}^* \rangle| &= \left|\sum_{i=1}^{m} x_i^*(x_i^{(n)})\right| + \left|\sum_{i=m+1}^{\infty} x_i^*(x_i^{(n)})\right| \\ &\le \sum_{i=1}^{m} |x_i^*(x_i^{(n)})| + |\langle \bar{x}^{(n)}, \bar{x}^*(i > m)\rangle| \\ &\le \varepsilon/2 + \|\bar{x}^{(n)}\|_{L^p_{\text{weak}}(X)} \cdot \|\bar{x}^*(i > m)\|_{L^{p'}\langle X^* \rangle} \\ &\le \varepsilon/2 + M \cdot \varepsilon/2M = \varepsilon. \end{aligned}$$

Therefore (4) follows. □

Similarly, we have

Lemma 8. *Let* $\bar{x}^{*(n)} = (x_i^{*(n)})_i \in L^{p'}_{\text{weak}}(X^*)$ *for each* $n \in \mathbf{N}$. *Then*

$$\sigma(L^{p'}_{\text{weak}}(X^*), L^p\langle X \rangle) - \lim_n \bar{x}^{*(n)} = 0 \tag{7}$$

if and only if

$$\sigma(X^*, X) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots \tag{8}$$

and

$$M = \sup_n \|\bar{x}^{*(n)}\|_{L^{p'}_{\text{weak}}(X^*)} < \infty. \tag{9}$$

Theorem 9. *Let* X *be a Banach space and* $1 < p < \infty$. *Then* $L^p[0,1]\hat{\otimes}X$, *the projective tensor product of* $L^p[0,1]$ *and* X, *is a Grothendieck space if and only if* X *is a Grothendieck space and each continuous linear operator from* $L^p[0,1]$ *to* X^* *is compact.*

Proof. By Propositions 2, 3 and 4, it is enough to show that $L^p\langle X\rangle$ is a Grothendieck pace if and only if X is a Grothendieck space and $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$.

Now suppose that X is a Grothendieck space and $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$. By Propositions 5 and 6,

$$(L^p\langle X\rangle)^* = L^{p'}_{\text{weak}}(X^*), \quad (L^p\langle X\rangle)^{**} = L^p\langle X^{**}\rangle. \tag{10}$$

Let $\bar{x}^{*(n)} = (x_i^{*(n)})_i \in L^{p'}_{\text{weak}}(X^*)$ be such that $\bar{x}^{*(n)}$ converges to 0 weak* in $(L^p\langle X\rangle)^*$, i.e.,

$$\sigma(L^{p'}_{\text{weak}}(X^*), L^p\langle X\rangle) - \lim_n \bar{x}^{*(n)} = 0.$$

By Lemma 8,

$$\sigma(X^*, X) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots$$

and

$$\sup_n \|\bar{x}^{*(n)}\|_{L^{p'}_{\text{weak}}(X^*)} < \infty.$$

Since X is a Grothendieck space,

$$\sigma(X^*, X^{**}) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots .$$

Note that $\bar{x}^{*(n)} \in L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$. By Lemma 7,

$$\sigma(L^{p'}_{\text{weak},0}(X^*), L^p\langle X^{**}\rangle) - \lim_n \bar{x}^{*(n)} = 0.$$

It follows from (10) that $\bar{x}^{*(n)}$ converges to 0 weakly in $(L^p\langle X\rangle)^*$, and hence, $L^p\langle X\rangle$ is a Grothendieck space.

On the other hand, suppose that $L^p\langle X\rangle$ is a Grothendieck space. It is obvious that X is a Grothendieck space. Next we want to show that $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$.

Let $\bar{x}^* = (x_i^*)_i \in L^{p'}_{\text{weak}}(X^*)$. For each $k \in \mathbf{N}$, define

$$\bar{x}^{*(k)} = (0, \dots, 0, x_k^*, 0, 0, \dots).$$

Then $\bar{x}^{*(k)} \in L^{p'}_{\text{weak}}(X^*)$ for each $k \in \mathbf{N}$. Next we want to show that the series $\sum_k \bar{x}^{*(k)}$ is subseries convergent series in $L^{p'}_{\text{weak}}(X^*) = (L^p\langle X\rangle)^*$.

For each fixed subsequence $n_1 < n_2 < \cdots$ and each $m \in \mathbf{N}$, define

$$\bar{z} = (\dots, x^*_{n_1}, \dots, x^*_{n_2}, \dots, x^*_{n_k}, \dots)$$

and

$$\bar{z}^{(m)} = \sum_{k=1}^{m} \bar{x}^{*(n_k)} = (\dots, x^*_{n_1}, \dots, x^*_{n_2}, \dots, x^*_{n_m}, 0, 0, \dots).$$

By Proposition 1, $\bar{z} \in L^{p'}_{\text{weak}}(X^*), \bar{z}^{(m)} \in L^{p'}_{\text{weak}}(X^*)$ for each $m \in \mathbf{N}$ and

$$\|\bar{z}^{(m)}\|_{L^{p'}_{\text{weak}}(X^*)} \le \|\bar{x}^*\|_{L^{p'}_{\text{weak}}(X^*)}, \quad m = 1, 2, \dots .$$

By Lemma 8,

$$\sigma(L^{p'}_{\text{weak}}(X^*), L^p\langle X\rangle) - \lim_m \bar{z}^{(m)} = \bar{z}.$$

Thus the partial sum $\sum_{k=1}^m \bar{x}^{*(n_k)}$ converges to $\bar{z}$ weak* in $(L^p\langle X\rangle)^*$. Since $L^p\langle X\rangle$ is a Grothendieck space, the partial sum $\sum_{k=1}^m \bar{x}^{*(n_k)}$ converges to $\bar{z}$ weakly in $(L^p\langle X\rangle)^*$. Hence we have shown that the series $\sum_k \bar{x}^{*(k)}$ is weakly subseries convergent in $(L^p\langle X\rangle)^*$. It follows from the Orlicz-Pettis theorem, cf. [**7**, p. 24], that the series $\sum_k \bar{x}^{*(k)}$ is subseries convergent in $(L^p\langle X\rangle)^*$, and hence, convergent in $(L^p\langle X\rangle)^*$. Therefore,

$$\lim_n \|\bar{x}^*(i > n)\|_{L^{p'}_{\text{weak}}(X^*)} = \lim_n \Big\| \sum_{k=n+1}^{\infty} \bar{x}^{*(k)} \Big\|_{(L^p\langle X\rangle)^*} = 0.$$

Thus $\bar{x}^* \in L^{p'}_{\text{weak},0}(X^*)$. □

Lemma 10. *Let* $\bar{x}^{(n)} = (x_i^{(n)})_i \in L^p\langle X\rangle$ *for each* $n \in \mathbf{N}$. *Then*

$$\sigma(L^p\langle X\rangle, L^{p'}_{\text{weak}}(X^*)) - \lim_n \bar{x}^{(n)} = 0 \tag{11}$$

is equivalent to

$$\sigma(X, X^*) - \lim_n x_i^{(n)} = 0, \quad i = 1, 2, \dots \tag{12}$$

and

$$M = \sup_n \|\bar{x}^{(n)}\|_{L^p\langle X\rangle} < \infty \tag{13}$$

if and only if $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$.

Proof. Suppose that $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$. Note that, for each $\bar{x}^* \in L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$, $\lim_n \|\bar{x}^*(i > n)\|_{L^{p'}_{\text{weak}}(X^*)} = 0$. Similarly as the proof of Lemma 7, we can show that (11) $\Leftrightarrow$ (12)+(13).

Now suppose that (11) $\Leftrightarrow$ (12) + (13). We want to show that $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$. If there exists an $\bar{x}^* = (x_i^*)_i \in L^{p'}_{\text{weak}}(X^*)$ but $\bar{x}^* \notin L^{p'}_{\text{weak},0}(X^*)$, then from Proposition 5,

$$\lim_n \|\bar{x}^*(i > n)\|_{L^{p'}_{\text{weak}}(X^*)} = \lim_n \sup \left\{ \left| \sum_{i=n+1}^{\infty} x_i^*(x_i) \right| : (x_i)_i \in B_{L^p\langle X\rangle} \right\} \neq 0.$$

Thus there are $\varepsilon_0 > 0$, $\bar{x}^{(k)} = (x_i^{(k)})_i \in B_{L^p\langle X\rangle}$, $k = 1, 2, \dots$ and a subsequence $n_1 < n_2 < \cdots$ such that

$$\left| \sum_{i=n_k}^{\infty} x_i^*(x_i^{(k)}) \right| \geq \varepsilon_0, \quad k = 1, 2, \dots .$$

Let $\bar{z}^{(k)} = (0, \dots, 0, x_{n_k}^{(k)}, x_{n_k+1}^{(k)}, \dots)$. Then $\bar{z}^{(k)} \in B_{L^p\langle X\rangle}$ for each $k \in \mathbf{N}$. Moreover, it is easy to see that

$$\sigma(X, X^*) - \lim_k z_i^{(k)} = 0, \quad i = 1, 2, \dots .$$

By hypothesis,

$$\sigma(L^p\langle X\rangle, L^{p'}_{\text{weak}}(X^*)) - \lim_k \bar{z}^{(k)} = 0.$$

But for each $k \in \mathbf{N}$,

$$|\langle \bar{z}^{(k)}, \bar{x}^* \rangle| = \left| \sum_{i=n_k}^{\infty} x_i^*(x_i^{(k)}) \right| \geq \varepsilon_0.$$

This contradiction shows that $L^{p'}_{\text{weak}}(X^*) = L^{p'}_{\text{weak},0}(X^*)$. □

Similarly we have

Lemma 11. *Let $\bar{x}^{*(n)} = (x_i^{*(n)})_i \in L^{p'}\langle X^* \rangle$ for each $n \in \mathbf{N}$. Then*

$$\sigma(L^{p'}\langle X^* \rangle, L^p_{\text{weak},0}(X)) - \lim_n \bar{x}^{*(n)} = 0 \tag{14}$$

if and only if

$$\sigma(X^*, X) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots \tag{15}$$

and

$$M = \sup_n \|\bar{x}^{*(n)}\|_{L^{p'}\langle X^* \rangle} < \infty. \tag{16}$$

Theorem 12. *Let X be a Banach space and $1 < p$, $p' < \infty$ such that $1/p + 1/p' = 1$. Then $L^p[0,1] \check{\otimes} X$, the injective tensor product of $L^p[0,1]$ and X, is a Grothendieck space if and only if X is a Grothendieck space and each continuous linear operator from $L^{p'}[0,1]$ to X^{**} is compact.*

Proof. By Propositions 3 and 4, it is enough to show that $L^p_{\text{weak},0}(X)$ is a Grothendieck space if and only if X is a Grothendieck space and $L^p_{\text{weak}}(X^{**}) = L^p_{\text{weak},0}(X^{**})$. By Propositions 5 and 6,

$$L^p_{\text{weak},0}(X)^* = L^{p'}\langle X^* \rangle, \qquad L^p_{\text{weak},0}(X)^{**} = L^p_{\text{weak}}(X^{**}). \tag{17}$$

Now suppose that X is a Grothendieck space and $L^p_{\text{weak}}(X^{**}) = L^p_{\text{weak},0}(X^{**})$. Let $\bar{x}^{*(n)} = (x_i^{*(n)})_i \in L^{p'}\langle X^* \rangle$ such that $\bar{x}^{*(n)}$ converges to 0 weak* in $L^p_{\text{weak},0}(X)^*$, i.e.,

$$\sigma(L^{p'}\langle X^* \rangle, L^p_{\text{weak},0}(X)) - \lim_n \bar{x}^{*(n)} = 0.$$

By Lemma 11,

$$\sigma(X^*, X) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots$$

and

$$\sup_n \|\bar{x}^{*(n)}\|_{L^{p'}\langle X^*\rangle} < \infty.$$

Since X is a Grothendieck space,

$$\sigma(X^*, X^{**}) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots .$$

Note that $L^p_{\text{weak}}(X^{**}) = L^p_{\text{weak},0}(X^{**})$. By Lemma 10,

$$\sigma(L^{p'}\langle X^*\rangle, L^p_{\text{weak}}(X^{**})) - \lim_n \bar{x}^{*(n)} = 0.$$

It follows from (17) that $\bar{x}^{*(n)}$ converges to 0 weakly in $L^p_{\text{weak},0}(X)^*$, and, hence, $L^p_{\text{weak},0}(X)$ is a Grothendieck space.

On the other hand, suppose that $L^p_{\text{weak},0}(X)$ is a Grothendieck space. It is obvious that X is a Grothendieck space. Next we want to show that $L^p_{\text{weak}}(X^{**}) = L^p_{\text{weak},0}(X^{**})$.

Let $\bar{x}^{*(n)} = (x_i^{*(n)})_i \in L^{p'}\langle X^*\rangle$ for each $n \in \mathbf{N}$ such that

$$\sigma(X^*, X^{**}) - \lim_n x_i^{*(n)} = 0, \quad i = 1, 2, \dots \tag{18}$$

and

$$\sup_n \|\bar{x}^{*(n)}\|_{L^{p'}\langle X^*\rangle} < \infty. \tag{19}$$

By Lemma 11,

$$\sigma(L^{p'}\langle X^*\rangle, L^p_{\text{weak},0}(X)) - \lim_n \bar{x}^{*(n)} = 0.$$

It follows from (17) that $\bar{x}^{*(n)}$ converges to 0 weak* in $L^p_{\text{weak},0}(X)^*$. Since $L^p_{\text{weak},0}(X)$ is a Grothendieck space, $\bar{x}^{*(n)}$ converges to 0 weakly in $L^p_{\text{weak},0}(X)^*$, i.e., from (17) again

$$\sigma(L^{p'}\langle X^*\rangle, L^p_{\text{weak}}(X^{**})) - \lim_n \bar{x}^{*(n)} = 0. \tag{20}$$

Thus we have shown that (18) + (19) $\Leftrightarrow$ (20). By Lemma 10, $L^p_{\text{weak}}(X^{**}) = L^p_{\text{weak},0}(X^{**})$. $\square$

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