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By G. EMMANUELE

Departimento di Matematica

Università di Catania

Catania 95125, Italy

The following is an exposition of still unpublished result concerning complemented copies of c_0 in Banach spaces. Our terminology will be standard. If a capital letter, like E, denotes a Banach space, then E^* denotes its dual space and $B(E)$ its unit ball.

The first result we state needs the concept of limited set. A (bounded) subset X of D is said to be limited ([3]) if for any weak* null sequence (x_n^*) in E^* we have

$$\lim_n \sup_X |x_n^*(x)| = 0$$

`Theorem 1.` *For a Banach space E the following are equivalent*

(i) E contains a complemented copy of c_0

(ii) there exists a basic sequence (x_n) equivalent to the unit vector basis of c_0 which is not limited.

`Remark.` More or less at the same time the previous result was obtained by T. Schlumprecht and it will appear in his Ph.D. Dissertation.

The above quoted result has the three following consequences

`Theorem 2.` *Let (S, Σ, μ) be a finite measure space and E be a Banach space containing a copy of c_0. Then the completion $\widehat{\mathcal{P}(\mu, E)}$ of the space of Pettis integrable functions with values in E (equipped with the norm $\|f\| = \sup\{\int_S |x^*(f(s))| \, d\mu : x^* \in B(E^*)\}$) contains a complemented copy of c_0.*

This result improves two similar ones obtained in [4], under certain restrictions on $\mathcal{P}(\mu, E)$.

`Theorem 3.` *Let E be a Banach space. Then $l^1(E)$ contains a complemented copy of c_0 iff E does.*

We observe that this result cannot be extended to $L^1(\mu, E)$ as it is shown in the paper [5] which will appear on Proc. Amer. Math. Soc.; in passing we note that Prof.

Drewnowski observed that the result in [5] can be extended without any trouble to Orlicz function spaces $L^{\phi}(\mu, E)$.

`Theorem 4.` *Let E be a Gelfand-Phillips space, i.e. a Banach space in which every limited set is relatively compact. If E contains a copy of c_0, it has to contain a complemented copy of c_0.*

Using a result from [1] as well as Theorem 4 we can also prove another result (see [1] for the definition of hereditary Dunford-Pettis property)

`Theorem 5.` *Let E be a Banach space. Then the following are equivalent*

(i) each weakly null sequence $(x_n) \subset E$ with $\inf_n \|x_n\| > 0$ has a subsequence $(x_{k(n)})$ equivalent to the unit vector basis of c_0 such that its closed linear span is complemented in E

(ii) E has the hereditary Dunford Pettis property and is a Gelfand Phillips space

Further consequences of the previous results are the following

`Corollary 6.` *Let E be a dual Banach space complemented in a Banach lattice. E is a Gelfand Phillips space iff $E \not\supset c_0$.*

we recall that if E has l.u.st. ([6]) or G.L.l.u.st. ([7]) then E^* is complemented in a Banach lattice.

In the following corollary Σ will be an algebra of subsets of an abstract set S and $fabv(\Sigma, E)$ will denote the usual Banach space of E-valued vector measures

`Corollary 7.` *Let E^* be complemented in a Banach lattice. Then $fabv(\Sigma, E^*)$ is a Gelfand Phillips space iff E^* is.*

The final results are consequences of the previous theorems; they concern with operators defined on suitable Banach spaces

`Corollary 8.` *Let E be a Banach lattice. The following are equivalent*

(i) E is a Grothendieck space (see [2])

(ii) any operator $T : E \to F, F$ a Gelfand-Phillips space, is weakly compact

(iii) any operator $T : E \to c_0$ is weakly compact.

`Corollary 9.` *Let E be a Banach space with the property (V) (see [8]). Then any strictly cosingular (see [9]) operator $T : E \to F, F$ a Gelfand-Phillips space, is weakly* *compact.*

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