The (BD) Property in $L^1(\mu,E)$

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A subset X of a Banach space E is *limited* if for each weak$*$ null sequence (x_n^*) in E^* one has

$$\lim_n \sup_X |x_n^*(x)| = 0;$$

E has the (BD) *property* if every limited set in E is relatively weakly compact. In this brief note we prove that the (BD) property lifts from E to $L^1(\mu,E)$, for any finite measure space (S,Σ,μ).

Bourgain and Diestel [2] showed that all spaces not containing ℓ^1 have the (BD) property; moreover, it is easy to see that the Gelfand–Phillips spaces [4] have this property as do spaces with the (RDP)* property [5]; in particular, it is easy to show that spaces with dual not containing ℓ^1, and weakly sequentially complete spaces too, have the (RDP)* property.

We recall that limited sets are bounded and conditionally weakly compact, that the image of a limited set under a linear operator is limited, and that a subset X of E is limited if and only if $T(X)$ is relatively compact in $c_\circ$ for $e =$ very linear operator $T{:}E \to c_\circ$ [2]. We also recall that a subset Y of $c_\circ$ is relatively compact if and only if $\lim_n \sup_Y |y_n| = 0$.

We refer the reader to [3, III.2] for the definitions of a finite partition π and the related conditional expectation E_π, and to [1] for the definition of $\Lambda_1(c_\circ)$; for our purpose it is sufficient to know that $\Lambda_1(c_\circ)$ is a set of operators from $L^1(\mu,E)$ to $c_\circ$.

Our result follows from a result on relative weak compactness in $L^1(\mu,E)$ due to Batt and Hiermeyer [1; 2.1, 2.6], which we state as a lemma.

Lemma. *Let X be a subset of $L^1(\mu,E)$ and suppose that:*

(i) *X is bounded and uniformly integrable,*

(ii) *for every $A \in \Sigma$ the set $X_A = \{\int_A f\,d\mu\colon f \in X\}$ is relatively weakly compact in E,*

(iii) *$T(X)$ is relatively weakly compact in $c_\circ$ for every T in $\Lambda_1(c_\circ)$,*

(iv) *for every increasing sequence (π_k) of finite partitions and every $L \in (L^1(\mu,E))^*$ one has*

$$\lim_n \sup_X |\langle L, E_{\pi_n} f - E_{\bigcup_k \pi_k} f\rangle| = 0.$$

Then X is relatively weakly compact in $L^1(\mu,E)$.

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Theorem. *Let E have the (BD) property. Then $L^1(\mu,E)$ also has the (BD) property.*

Proof. Let X be a limited subset of $L^1(\mu,E)$. Then, since X is bounded and conditionally weakly compact, X must be uniformly integrable [3, IV.2.4].

Further, X_A is the image of X under the linear operator

$$T_A\colon f \to \int_A f\,d\mu;$$

X_A is therefore limited in E and so it is relatively weakly compact (for any $A \in \Sigma$).

Also $T(X)$ is relatively compact in $c_\circ$ for *any* linear operator $T\colon L^1(\mu,E) \to c_\circ$; *a fortiori* for any $T \in \Lambda_1(c_\circ)$.

Let L belong to $(L^1(\mu,E))^*$. Since

$$\|E_{\pi_n} f - E_{\bigcup\limits_k \pi_k} f\| \to 0,$$

we can again define the operator

$$T_{(\pi_k),L}\colon f \to (\langle L, E_{\pi_n} f - E_{\bigcup\limits_k \pi_k} f\rangle)$$

from $L^1(\mu,E)$ to $c_\circ$; and since $T_{(\pi_k),L}(X)$ is relatively compact, we see that

$$\lim_n \sup_X |\langle L, E_{\pi_n} f - E_{\bigcup\limits_k \pi_k} f\rangle| = 0.$$

The lemma now assures us that X is relatively weakly compact; the theorem is proved. □

References

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