About the position of $K_{w^*}(E^*, F)$ inside $L_{w^*}(E^*, F)$

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About the Position of $K_{w^*}(E^*, F)$ Inside $L_{w^*}(E^*, F)$ (**).

Abstract. – *We prove that if* c_0 *embeds into* $K_{w^*}(E^*, F)$ *then either* $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ *or* $K_{w^*}(E^*, F)$ *is uncomplemented in the space* $L_{w^*}(E^*, F)$. *A lot of consequences of this result are then presented; among them some theorems about the position of* $K(E, F)$ *inside* $W(E, F)$ *and of* $\mathrm{cca}(\Sigma, E)$ *inside* $\mathrm{ca}(\Sigma, E)$.

Introduction.

Let E, F be two Banach spaces. An old, not yet solved, conjecture about $K(E, F)$ (= space of compact operators from E into F) and $L(E, F)$ (= space of bounded linear operators from E into F) is the following: *either* $K(E, F) = L(E, F)$ *or* $K(E, F)$ *is uncomplemented in* $L(E, F)$. The best result we know seems to be the following theorem due to Feder ([8])

THEOREM 1 ([8]). *Let us assume there exist* $T \in L(E, F) \setminus K(E, F)$ *and a sequence* (T_n) *in* $K(E, F)$, *such that for any* $x \in E$ $\Sigma T_n(x) = T(x)$ *unconditionally. Then* $K(E, F)$ *is uncomplemented in* $L(E, F)$.

Recently ([6]) we proved that Feder hypothesis is exactly equivalent to the existence of copies of c_0 inside $K(E, F)$. So the previous result can be reformulated as it follows.

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THEOREM 2 ([6]). *Let c_0 embed into $K(E,F)$. Then $K(E,F)$ is uncomplemented in $L(E,F)$.*

In this note we want to extend Theorem 2 above to a different situation, in some sense improving the same Theorem 2 (see Corollary 7 in section 2). Precisely, let $K_{w^*}(E^*, F)$ denote the space of compact, w^*-w continuous operators from E^* into F; we prove (with techniques completely different from those employed in [8]) the following

MAIN THEOREM. *If c_0 embeds into $K_{w^*}(E^*, F)$ then either $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ (= space of w^*-w continuous operators from E^* into F) or $K_{w^*}(E^*, F)$ is uncomplemented in $L_{w^*}(E^*, F)$.*

In passing we observe that we even show that, if c_0 embeds into $K_{w^*}(E^*, F)$, then $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ if and only if either E or F has the Schur property. Since $K(E, F) \simeq K_{w^*}(E^{**}, F)$ and $W(E, F) \simeq L_{w^*}(E^{**}, F)$ ([13]) ($W(E, F)$ = space of weakly compact operators from E into F), our Main Theorem improves Theorems 1 and 2 (at least in the case of E^* and F without the Schur property). Furthermore, in the case $F = \mathrm{ca}(\Sigma)$ (= space of countably additive measures from a σ-algebra Σ into R) we get results about $\mathrm{ca}(\Sigma, E)$ (= space of countably additive vector measures). To this purpose we recall that $\mathrm{cca}(\Sigma, E) \simeq K_{w^*}(E^*, \mathrm{ca}(\Sigma))$ and that $\mathrm{ca}(\Sigma, E) \simeq L_{w^*}(E^*, \mathrm{ca}(\Sigma))$ ([13]). We note that in all the considered spaces of operators and vector measures the norm will be the sup norm.

1. – The Main Theorem.

The first section is devoted to a proof of the Main Theorem of the paper (Theorem 4 below), already quoted in the Introduction, about the position of $K_{w^*}(E^*, F)$ inside $L_{w^*}(E^*, F)$. We first need a Lemma about $K(l_1, F)$ and $W(l_1, F)$

LEMMA 3. *Let F be a Banach space without the Schur property. Then $K(l_1, F)$ is uncomplemented in $W(l_1, F)$.*

PROOF. Our proof is a suitable modification of that of Lemma 3 in [9] (and the present lemma actually is an improvement of that result). Let (y_n) be a normalized w-null basic sequence in F. Define, for $\xi \in l_\infty$, a linear, bounded operator $\Phi(\xi)$ by putting

$$\Phi(\xi)(g) = \sum \xi_n e_n(g) y_n, \qquad g \in l_1$$

(where (e_n) is the unit vector basis of c_0 in l_∞). Note that the series converges absolutely and $\Phi(\xi) \in W(l_1, F)$ since $y_n \xrightarrow{w} \theta$. Furthermore, the map $\xi \to \Phi(\xi)$ is linear, bounded from l_∞ into $W(l_1, F)$. If (y_n^*) is a bounded sequence of biorthogonal functionals for (y_n) we can define $R: W(l_1, F) \to W(l_1, l_\infty)$ (with $R(K(l_1, F) \subset K(l_1, l_\infty))$ by putting

$$R(T)(g) = (T(g), y_n^*), \qquad g \in l_1 .$$

Let us assume $P: W(l_1, F) \xrightarrow{\text{onto}} K(l_1, F)$ is a projection. We consider the following two linear, bounded operators

$$R\Phi,\ RP\Phi: l_\infty \to K(l_1, l_\infty)$$

for which we clearly have

$$R(\Phi(\xi)) = R(P\Phi(\xi)), \qquad \xi \in c_0 .$$

Proposition 5 in [9] gives the existence of an infinite subset M of N such that

$$R(\Phi(\xi)) = R(P\Phi(\xi)), \qquad \xi \in l_\infty(M) .$$

In particular, we have

$$R(\Phi(1_M)) = R(P\Phi(1_M)),$$

where 1_M equals 1 if $n \in M$, 0 otherwise. Hence

$$R(P\Phi(1_M))(g) = \left(\sum_{h \in M} e_h(g) y_h,\ y_n^*\right), \qquad g \in l_1 .$$

On the other hand, $R(P\Phi(1_M))$ is compact. This means that the sequence $(R(P\Phi(1_M))(e_n^*))$, (e_n^*) the unit vector basis of l_1, must be relatively compact in l_∞. But, for all $m \in M$, we have

$$R(P\Phi(1_M))(e_n^*) = (y_m, y_n^*) = e_m$$

a non-relatively compact subset of l_∞. This contradiction proves that $K(l_1, F)$ is not complemented in $W(l_1, F)$.

We are now ready for the main Theorem, in which we even use the existence of the following isometric isomorphisms: $K_{w^*}(E^*, F) \simeq K_{w^*}(F^*, E)$ and $L_{w^*}(E^*, F) \simeq L_{w^*}(F^*, E)$ (via the mapping $T \to T^*$)

THEOREM 4. *Let c_0 embed into $K_{w^*}(E^*, F)$. Then either $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ or $K_{w^*}(E^*, F)$ is uncomplemented in $L_{w^*}(E^*, F)$. Furthermore, $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ if and only if one of the following two mutually exclusive facts is verified*

i) *c_0 embeds into E and F has the Schur property*

ii) *c_0 embeds into F and E has the Schur property.*

PROOF. Let us suppose $K_{w^*}(E^*, F) \neq L_{w^*}(E^*, F)$. This means that E and F do not possess the Schur property. Let l_∞ embed into F. It is well known that l_∞ is then complemented in F. If P from $L_{w^*}(E^*, F)$ onto $K_{w^*}(E^*, F)$ were a projection, it can be easily seen that $K_{w^*}(E^*, l_\infty)$ should be complemented in $L_{w^*}(E^*, l_\infty)$. Now, observe that $L_{w^*}(E^*, l_\infty) = L_{w^*}(l_\infty^*, E) = W(l_1, E)$ and that $K_{w^*}(E^*, l_\infty) = K_{w^*}(l_\infty^*, E) = K(l_1, E)$. Hence $K(l_1, E)$ should be complemented in $W(l_1, E)$; since E does not have the Schur property, we obtain something contradicting Lemma 3. So we can suppose (and we do) that F does not contain l_∞ (and even that E does not contain l_∞ since $L_{w^*}(E^*, F) \simeq L_{w^*}(F^*, E)$ and $K_{w^*}(E^*, F) \simeq K_{w^*}(F^*, E)$). A recent result of Drewnowski ([3]) gives that l_∞ does not live inside of $K_{w^*}(F^*, E)$. Now, let us assume F contains a copy of c_0 (the same proof works if E contains a copy of c_0). Let (x_n) be a normalized, w-null basic sequence in E, (y_n) be a copy of the unit vector basis of c_0 in F and define, for $\xi \in l_\infty$, a linear, bounded operator by putting

$$\Phi(\xi)(x^*) = \sum \xi_n x_n(x^*) y_n, \qquad x^* \in E^*$$

(note that the series converges unconditionally). We show that $\Phi(\xi) \in L_{w^*}(E^*, F)$. To this aim, it will be enough to consider a w^*-null net $(x_\alpha^*) \subset B_{E^*}$ and $y^* \in B_{F^*}$ and to prove that

$$\lim_\alpha |\Phi(\xi)(x_\alpha^*)(y^*)| = 0 . \tag{1}$$

Since $\sum |y_n(y^*)| < +\infty$ given $\gamma > 0$ there is $p \in N$ such that

$$\sum_{n=p+1}^{\infty} |y_n(y^*)| < \gamma/2\|\xi\|$$

(if $\|\xi\| = 0$ there is nothing to prove). On the other hand, we have

$$\lim_\alpha \sum_{n=1}^{p} |\xi_n x_n(x_\alpha^*) y_n(y^*)| = 0 .$$

These two facts together give (1). Hence $\Phi(\xi) \in L_{w^*}(E^*, F)$. Let us suppose $P: L_{w^*}(E^*, F) \xrightarrow{\text{onto}} K_{w^*}(E^*, F)$ is a projection. Since the map $\xi \to \Phi(\xi)$ is easily seen to be linear, bounded from l_∞ into $L_{w^*}(E^*, F)$ we get a linear, bounded operator $P\Phi$ from l_∞ into $K_{w^*}(E^*, F)$. Since l_∞ does not embed into $K_{w^*}(E^*, F)$, $P\Phi$ is weakly compact ([12]) and so $P\Phi(e_n) \to \theta$; but it is clear that $P\Phi(e_n) = x_n \otimes y_n \not\to \theta$. The reached contradiction concludes the proof when c_0 embeds into F (or E). So we can assume that c_0 lives inside $K_{w^*}(E^*, F)$ but not in E and in F. Let (T_n) be a copy of the unit vector basis of c_0 in $K_{w^*}(E^*, F)$. For $\xi \in l_\infty$ define (as above) a linear, bounded operator

$$\Phi(\xi)(x^*) = \sum \xi_n T_n(x^*), \qquad x^* \in E^* .$$

The series converges unconditionally, because $\sum T_n$ is weakly unconditionally converging and c_0 does not live inside F. The map $\xi \to \Phi(\xi)$ is linear, bounded from l_∞ into $L(E^*, F)$; but actually, $\Phi(\xi) \in L_{w^*}(E^*, F)$; the proof of this last assertion is similar to the previous one (case of c_0 embedded into F) once we note that $\sum T_n^*(y^*)$ converges unconditionally in E since E does not contain c_0. Assuming the existence of a projection P from $L_{w^*}(E^*, F)$ onto $K_{w^*}(E^*, F)$, we can conclude our proof as in the previous case. It remains to show just the second claim of our statement. We first observe that if i) or ii) is true, then $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ (and we note that this equality holds true as soon as either E or F has the Schur property, without any other hypothesis). Now, look at the proof of the final part of the first statement: if c_0 embeds into $K_{w^*}(E^*, F)$ but not in E and F, then $K_{w^*}(E^*, F)$ must be uncomplemented in $L_{w^*}(E^*, F)$. Hence, if $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$, then either E or F has to contain c_0. Let us assume E does it. Let (y_n) be a w-null sequence in F. Define, as above, an element of $L_{w^*}(E^*, F)$ by putting

$$T(x^*) = \sum x_n(x^*) y_n, \qquad x^* \in E^*$$

where (x_n) is a copy of the unit vector basis of c_0 in E. T must be compact and so (y_n) is forced to be norm null. We are done.

THEOREM 5. *Let c_0 embeds into $K_{w^*}(E^*, F)$. Then l_∞ embeds into $L_{w^*}(E^*, F)$, provided E and F do not have the Schur property.*

PROOF. If c_0 embeds into E or F, we can define $\Phi: l_\infty \to L_{w^*}(E^*, F)$ as in Theorem 4. Since $\Phi(e_n) \not\to \theta$, there is an infinite subset M of N such that $\Phi|_{l_\infty(M)}$ is an isomorphism (see [12]). But $l_\infty(M)$ is isomorphic to l_∞. When c_0 embeds into $K_{w^*}(E^*, F)$ but not in E and F we can repeat a similar construction. We are done.

The next result is similar to Theorem 6 in [9]

THEOREM 6. *Let F have an unconditional finite dimensional expansion (A_n) of identity (we refer to* [9] *for this notion and equivalent reformulations). Then the following are equivalent*

i) $K_{w^*}(E^*, F) \neq L_{w^*}(E^*, F)$,

ii) *E and F do not possess the Schur property and c_0 embeds into* $K_{w^*}(E^*, F)$,

iii) *E and F do not possess the Schur property and l_∞ embeds into* $L_{w^*}(E^*, F)$,

iv) $K_{w^*}(E^*, F)$ *is uncomplemented in* $L_{w^*}(E^*, F)$.

PROOF. ii) implies iii) is in Theorem 5 and ii) implies iv) is in Theorem 4. iii) implies i) follows from Theorem 4, too; indeed, if $K_{w^*}(E^*, F) = L_{w^*}(E^*, F)$ under iii), the second part of Theorem 4 gives a contradiction. iv) implies i) is trivial. So it remains only to show that i) implies ii). It is clear that E and F are not allowed to possess the Schur property. Let $T \in L_{w^*}(E^*, F) \setminus K_{w^*}(E^*, F)$; the series $\sum A_n T$ verifies the assumptions contained in Theorem 1. The proof of Theorem 2 from [6] shows that c_0 must be inside $K_{w^*}(E^*, F)$. We are done.

2. - Applications of the Main Theorem to $K(E, F)$ and $W(E, F)$.

As remarked at the beginning, we have the following isometric isomorphisms $K(E, F) \simeq K_{w^*}(E^{**}, F)$ and $W(E, F) \simeq L_{w^*}(E^{**}, F)$. Hence, from the results in section 1 the following corollaries follow immediately

COROLLARY 7. *Let c_0 embed into $K(E, F)$. Then either $K(E, F) = W(E, F)$ or $K(E, F)$ is uncomplemented in $W(E, F)$. Furthermore, $K(E, F) = W(E, F)$ if and only if one of the following two mutually exclusive facts is true*

i) *c_0 embeds into E^* and F has the Schur property,*

ii) *c_0 embeds into F and E^* has the Schur property.*

COROLLARY 8. *Let c_0 embeds into $K(E, F)$. Then l_∞ embeds into $W(E, F)$, provided neither E^* nor F has the Schur property.*

COROLLARY 9. *Let either E^* or F have an unconditional finite dimensional expansion of the identity. Then the following are equivalent*

i) $K(E, F) \neq W(E, F)$,

ii) *E^* and F do not possess the Schur property and c_0 embeds into $K(E, F)$,*

iii) *E^* and F do not posses the Schur property and l_∞ embeds into $W(E, F)$,*

iv) *$K(E, F)$ is uncomplemented in $W(E, F)$.*

The results of section 1 also have other interesting consequences; the first two improve results by Kalton ([9]) and Feder ([7]) about $K(E, F)$ and $L(E, F)$

COROLLARY 10. *Let E have an unconditional finite dimensional expansion of the identity. Then the same conclusion of Corollary 9 is true.*

PROOF. As a consequence of Corollaries 7 and 8 we have just to show that i) implies ii). Under i) E^* and F are not allowed to possess the Schur property; furthermore, i) implies that $K(E, F) \neq L(E, F)$ and so we have just to appeal to Theorem 6 in [9] to get a copy of c_0 inside $K(E, F)$. We are done.

Corollary 11 below follows from Corollaries 7 and 8 and the result in [7] as Corollary 10 from Corollaries 7 and 8 and the result in [9] (so we do not give the proof of Corollary 11)

COROLLARY 11. *Let us assume one of the following hypotheses is verified*

1) *E is weakly compactly generated, F is a subspace of a space G with a shrinking unconditional basis and E^* or F^* has the bounded approximation property,*

2) *E is a quotient of a space G having a shrinking unconditional basis and either E^* has the bounded approximation property or F^* is separable and has the bounded approximation property.*

Then the same conclusion of Corollary 9 is true.

We observe that Corollaries 7-11 are improvements of the corresponding Theorems in the papers [3], [7], [8], [9] when E^* and F do not possess the Schur property. The following result seems to be new

COROLLARY 12. *Let us assume that E has the Dunford-Pettis property and there is a linear, bounded operator $T\colon l_2 \to F$ taking the unit* *vector basis* (e_n^2) *of* l_2 *onto a normalized basic sequence. Then the same conclusion of Corollary 9 is true.*

PROOF. As in Corollary 10 we have just to prove that i) implies ii). First, note that under i) E^* and F do not possess the Schur property (for F this is true as a consequence of our assumption about the existence of a special $T: l_2 \to F$). On the other hand, E has the Dunford-Pettis property and hence it must contain l_1 ([1]). It is known that this implies that L_1 must live inside E^* ([4]) and hence l_2 must do the same. The sequence $(e_n^2 \otimes T(e_n^2))$ is a copy of the unit vector basis of c_0 inside $K(E, F)$ (see [6]). We are done.

3. – Applications of the Main Theorem to cca(Σ, E) and ca(Σ, E).

In the Introduction we observed that $\mathrm{cca}(\Sigma, E) \simeq K_{w^*}(E^*, \mathrm{ca}(\Sigma))$ and $\mathrm{ca}(\Sigma, E) \simeq L_{w^*}(E^*, \mathrm{ca}(\Sigma))$. Hence, the results of section 1 have corollaries similar to Corollaries 7 and 9.

COROLLARY 13. *Let* c_0 *embed into* $\mathrm{cca}(\Sigma, E)$. *Then either* $\mathrm{cca}(\Sigma, E) = \mathrm{ca}(\Sigma, E)$ *or* $\mathrm{cca}(\Sigma, E)$ *is uncomplemented in* $\mathrm{ca}(\Sigma, E)$. *Furthermore,* $\mathrm{cca}(\Sigma, E) = \mathrm{ca}(\Sigma, E)$ *if and only if* c_0 *embeds into E and* $\mathrm{ca}(\Sigma)$ *has the Schur property (we recall* $\mathrm{ca}(\Sigma)$ *has the Schur property if and only if every finite positive measure on* Σ *is purely atomic).*

COROLLARY 14. *Let E have an unconditional finite dimensional expansion of the identity. Then the following are equivalent*

i) $\mathrm{cca}(\Sigma, E) \neq \mathrm{ca}(\Sigma, E)$,

ii) *E and* $\mathrm{ca}(\Sigma)$ *do not possess the Schur property and* c_0 *embeds into* $\mathrm{cca}(\Sigma, E)$,

iii) *E and* $\mathrm{ca}(\Sigma)$ *do not possess the Schur property and* l_∞ *embeds into* $\mathrm{ca}(\Sigma, E)$,

iv) $\mathrm{cca}(\Sigma, E)$ *is uncomplemented in* $\mathrm{ca}(\Sigma, E)$.

Now, we want to point out some other results about these two spaces. Recently, Drewnowski ([2]) has shown the following

THEOREM 15 ([2]). *Suppose the* σ*-algebra* Σ *admits a nonzero atomless finite positive measure. Then the following facts are equivalent*

a) $\mathrm{ca}(\Sigma, E) \supset l_\infty$,

b) there exists a noncompact operator $T: l_2 \to E$,

c) $\mathrm{ca}(\Sigma, E) \supset c_0$.

From now on, we assume the validity of Drewnowski's hypothesis. We can improve his Theorem by remarking that the following result is true

THEOREM 16. *Suppose the σ-algebra Σ admits a nonzero atomless finite positive measure. Then b) of Theorem 15 implies*

c') $\mathrm{cca}(\Sigma, E)$ *contains* c_0.

PROOF. Let $T: l_2 \to E$ as in *b*). T^* is a not compact operator from E^* into l_2. If P_n denotes the n-th projection in l_2, the operator $P_n T^*$ is in $K_{w^*}(E^*, l_2)$. Let j be an isomorphic embedding of l_2 into $\mathrm{ca}(\Sigma)$. We have that $jP_n T^* \in K_{w^*}(E^*, \mathrm{ca}(\Sigma))$ and $jT^* \in L_{w^*}(E^*, \mathrm{ca}(\Sigma))$. Furthermore, the hypotheses of Theorem 1 are verified (with the obvious changes) and so following the proof of Theorem 2 (see [6]) we reach our goal. We are done.

As observed before, *c')* gives that $\mathrm{cca}(\Sigma, E)$ is not complemented in $\mathrm{ca}(\Sigma, E)$; and so the existence of a non compact operator T from l_2 into E implies that $\mathrm{cca}(\Sigma, E)$ is not complemented in $\mathrm{ca}(\Sigma, E)$. This remark, Corollary 4 and Theorem 15 show that $\mathrm{cca}(\Sigma, l_p) = \mathrm{ca}(\Sigma, l_p)$ if $1 \leq p < 2$ and $\mathrm{cca}(\Sigma, l_p)$ is uncomplemented in $\mathrm{ca}(\Sigma, l_p)$ if $2 \leq p \leq +\infty$, this way improving results from [2]. Furthermore, we have the following new

THEOREM 17. *Let E be the dual of a space Y with the Dunford-Pettis property. Then either* $\mathrm{cca}(\Sigma, E) = \mathrm{ca}(\Sigma, E)$ *or* $\mathrm{cca}(\Sigma, E)$ *is uncomplemented in* $\mathrm{ca}(\Sigma, E)$.

PROOF. If $\mathrm{cca}(\Sigma, E) \neq \mathrm{ca}(\Sigma, E)$, then E does not enjoy the Schur property. As in Corollary 12, l_2 embeds into E. Hence *b*) of Theorem 15 is true. The above remarks conclude the proof.

We also observe that if $T \in L_{w^*}(E^*, \mathrm{ca}(\Sigma))$, which implies it is a weakly compact operator, its range is contained in a suitable $L_1(\mu)$ space. And so if such a T is not compact (*i.e.* if $\mathrm{cca}(\Sigma, E) \simeq \simeq K_{w^*}(E^*, \mathrm{ca}(\Sigma)) \neq L_{w^*}(E^*, \mathrm{ca}(\Sigma)) \simeq \mathrm{ca}(\Sigma, E)$) and moreover if it factorizes through l_2, *b*) of Theorem 15 is true. For hypotheses that guarantee the existence of such a factorization we refer to [10]. The last result we want to present implies that *c')* of Theorem 16 can be improved by the following:

c'') $\mathrm{cca}(\Sigma, E)$ *contains a complemented copy of* c_0.

We first need the definition of Gel'fand-Phillips space: a Banach space X is a *Gel'fand-Phillips space* if any *limited* subset M of X is relatively compact (a bounded subset is limited if for any w^*-null sequence (x_n^*) in X^* one has $\lim_n \sup_M |x_n^*(x)| = 0$). It is known that $\mathrm{ca}(\Sigma)$ is a Gel'fand-Phillips space ([14]).

THEOREM 18. *Let c_0 embed into $K_{w^*}(E^*, F)$. If F is a Gel'fand-Phillips space, then c_0 embeds complementably into $K_{w^*}(E^*, F)$.*

PROOF. If c_0 embeds into either E or F, the thesis is true by virtue of a result due to Ryan ([11]). So let us suppose c_0 does not live inside E and F. Let (T_n) be a copy of the unit vector basis of c_0 in $K_{w^*}(E^*, F)$. Take $(x_n^*) \subset B_{E^*}$ with $\inf_n \|T_n(x_n^*)\| > 0$.

We affirm that $T_n(x_n^*) \xrightarrow{w} \theta$; indeed, if $y^* \in F^*$ we have

$$|T_n(x_n^*)(y^*)| \leq \|T_n^*(y^*)\| \to 0$$

because $\sum T_n^*(y^*)$ is unconditionally converging in E (recall that E does not contain c_0). This implies that $(T_n(x_n^*))$ is not limited in F and so there is a w^*-null sequence $(y_n^*) \subset B_{F^*}$ for which $\inf_n |T_n(x_n^*)(y_n^*)| > 0$. Now, observe that the sequence $(x_n^* \otimes y_n^*)$ is w^*-null in $(K_{w^*}(E^*, F))^*$; indeed, for $T \in K_{w^*}(E^*, F)$ we have

$$|\langle T, x_n^* \otimes y_n^* \rangle| \leq \|T^*(y_n^*)\| \to 0 .$$

This means that (T_n) is not limited in $K_{w^*}(E^*, F)$. A result from [5] and [14] gives that a suitable subsequence of (T_n) spans a complemented copy of c_0 inside $K_{w^*}(E^*, F)$ as required. We are done.

We remark that if F has an unconditional finite dimensional expansion of the identity, then it is a Gel'fand-Phillips space. And so Theorem 6 can be improved, because ii) is equivalent to

ii') *E and F do not possess the Schur property and c_0 embeds complementably into $K_{w^*}(E^*, F)$.*

Similar improvements can be carried out in Corollary 10 and Corollary 11 (under (1)).

At the end, we observe that Theorem 18 has the following consequence concerning a special (not closed) subspace of $\mathrm{cca}(\Sigma, E)$

COROLLARY 19. *Let (S, Σ, ν) be a finite measure space. If c_0 embeds into $P_c(\nu, E)$ (= normed space of Pettis integrable functions with indefinite integral having relatively compact range), then it actually embeds complementably into $P_c(\nu, E)$.*

PROOF. It is known that $P_c(\nu, E)$ is a subspace of $K_{w^*}(E^*, L_1(\nu))$ and $L_1(\nu)$ is a Gel'fand-Phillips space. If E contains a copy of c_0 the result is true as it was showed by J. Diestel (see note after the present Corollary). If E does not contain c_0, we can use the same proof of Theorem 18. We are done.

Corollary 19 improves a result due to J. Diestel (unpublished, 1988) who proved that if c_0 embeds into E, then it embeds complementably into $P_c(\nu, E)$.

NOTE. The results contained in the last two sections were the content of the talk given by the author at the II Congreso de Analisis Funcional; Jarandilla de la Vera (Càceres, España), 18-22 Junio 1990.

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