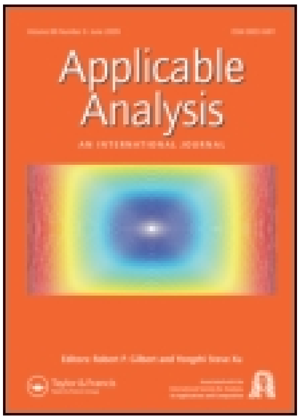

Applicable Analysis: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gapa20

Integrable solutions of hammerstein

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Version of record first published: 20 Jan 2011.

Applicable Analysis, Vol.50, pp. 277–284
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Integrable Solutions of Hammerstein Integral Equations

Communicated by M. Z. Nashed

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Abstract. We consider a Hammerstein integral equation and we prove that it has at least a solution in a suitable subset of $L^1[0,1]$ under quite general assumptions. This result has natural extensions to the case of $L^p[0,1]$ and to the case of finite dimensional spaces. At the end, another result about existence of integrable solutions is presented, too.

KEY WORDS: Hammerstein nonlinear integral equation

(Received for Publication 22 October 1990)

One of the most investigated integral equations in nonlinear functional analysis is the Hammerstein equation

(1) $$x(t)=\varphi(t)+\int_0^1 k(t,s)f(s,x(s))ds \qquad t\in[0,1].$$

It has been studied in several papers and monographs ([1], [2], [3], [4], [8], [9], [12]), and existence results have been obtained under several different groups of hypotheses; most of these results requires rather strong assumptions like coercivity, monotonicity, differentiability on k and f.

A quite general result has been obtained recently in [3]; the author of [3] was however forced, by the technique he used, to consider some monotonicity assumptions on φ and k. In this short note we are able to dispense with these hypotheses. Our proof makes use of the Schauder fixed point Theorem as in [3], but we look-for a solution in a different subset of $L^1[0,1]$ and this allows us to avoid the monotonicity hypotheses considered in [3]; we however need to suppose that k is nonnegative, whereas in [3] k is allowed to assume values in R. It is easy to see that our proof again works if one assumes "$k:[0,1]\times[0,1]\to R$ and the operator K is regular" (see [12] for this definition); we leave to the reader the proof of this fact.

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(1) Work performed under the auspices of G.N.A.F.A. of C.N.R. and partially supported by M.U.R.S.T. of Italy (60%;1987)

Throughout, we shall assume the following four hypotheses

(h_1) $\varphi \in L^1[0,1]$

(h_2) $f:[0,1]\times R \longrightarrow R$ *verifies Caratheodory hypotheses, i.e.* f *is measurable with respect to* $t\in[0,1]$, *for all* $x\in R$, *and continuous with respect to* $x\in R$, *for almost all* $t\in[0,1]$, *and moreover there exist* $a\in L^1[0,1]$ *and* $b\geq 0$ *such that*

$$|f(t,x)|\leq a(t)+b|x| \quad \text{for a.a. } t\in[0,1] \text{ and all } x\in R$$

(h_3) $k:[0,1]\times[0,1]\longrightarrow R_+$ *is measurable with respect to both variables and is such that the integral operator*

$$(Kx)(t)=\int_0^1 k(t,s)x(s))ds \qquad t\in[0,1]$$

maps $L^1[0,1]$ *into itself.*

We recall that under (h_2) the operator

$$(Fx)(t)=f(t,x(t)) \qquad t\in[0,1]$$

maps $L^1[0,1]$ into itself continuously (see [7] where actually is proved that (h_2) is both a necessary and sufficient condition) and that under (h_3) the linear operator K maps $L^1[0,1]$ into itself continuously ([12]). Let $\|K\|$ denote the operator norm of K. We shall also assume

(h_4) $b\|K\|<1$.

Before presenting our result we need to recall two well known results about measurable functions

LUSIN THEOREM ([5]). *Let* $\varphi:[0,1]\longrightarrow R$ *be a measurable function. For any* $\epsilon>0$ *there is a closed subset* A_ϵ *of* $[0,1]$, $m(A_\epsilon^c)<\epsilon$, *such that* φ *restricted to* A_ϵ *is continuous.*

SCORZA DRAGONI THEOREM ([11]). *Let* $k:[0,1]\times[0,1]\longrightarrow R$ *be a function verifying Caratheodory hypotheses (see* (h_2)*). For any* $\epsilon>0$ *there is a closed subset* A_ϵ *of* $[0,1]$, $m(A_\epsilon^c)<\epsilon$, *such that* k *restricted to* $A_\epsilon\times[0,1]$ *is continuous.*

The main result will make use of the following Lemma

LEMMA. *Let us assume* $(h_1),(h_2),(h_3),(h_4)$. *Then there exists a unique, a.e. nonnegative, function* x_0, $x_0\in L^1[0,1]$, *such that*

$$(2)\qquad x_0(t)=|\varphi(t)|+\int_0^1 k(t,s)(a(s)+bx_0(s))ds\ ,\qquad t\in[0,1]\ .$$

Proof. Let us put $B_r=\{x\in L^1[0,1],\ \|x\|\le r\}$ where $r=\|\psi\|/(1-b\|K\|)$ with $\psi(\cdot)=|\varphi(\cdot)|+\int_0^1 k(\cdot,s)a(s)ds$. We consider the operator $A:L^1[0,1]\longrightarrow$ $\longrightarrow L^1[0,1]$ defined by

$$Ax(t)=|\varphi(t)|+\int_0^1 k(t,s)(a(s)+bx(s))ds\ ,$$

and we show, first, that $A(B_r)\subset B_r$. Indeed, for $x\in B_r$ we have

$$\|Ax\|=\int_0^1 |Ax(t)|dt\le\int_0^1 |\varphi(t)|dt+\int_0^1\left|\int_0^1 k(t,s)(a(s)+bx(s))ds\right|dt\le$$

$$\le\|\psi\|+b\int_0^1\left|\int_0^1 k(t,s)x(s)ds\right|dt\le\|\psi\|+b\|K\|\,\|x\|\le$$

$$\le\|\psi\|+b\|K\|\ \frac{\|\psi\|}{1-b\|K\|}\ =r\ .$$

If we consider $B_r^+=\{x:\ x\in B_r,\ x(t)\ge0\ \text{a.e.}\}$ we clearly have that $A(B_r^+)\subset B_r^+$. Furthermore, B_r^+ is a closed subset of $L^1[0,1]$ and so it is a complete metric space. We shall prove that A is a contraction and so our thesis will follow. Let $x_1,x_2\in B_r^+$. We have

$$\|Ax_1-Ax_2\|=\int_0^1\left|\int_0^1 k(t,s)b(x_1(s)-x_2(s))ds\right|dt\le b\|K\|\,\|x_1-x_2\|$$

where $b\|K\|<1$ thanks to (h_4). We are done.

THEOREM 1. *Let us assume* $(h_1),(h_2),(h_3),(h_4)$ *and the following*
(h_5) k *satisfies Caratheodory hypotheses, i.e. it is measurable with respect to* $t\in[0,1]$, *for all* $s\in[0,1]$, *and continuous with respect to* $s\in[0,1]$, *for almost all* $t\in[0,1]$.

Then the equation (1) *has a solution in* $L^1[0,1]$.

Proof. Let x_0 be the function verifying (2) in the Lemma. First of all, assume $x_0=\theta_{L^1[0,1]}$. In this case, we have, for $y(t)=\varphi(t)+\int_0^1 k(t,s)f(s,x_0(s))ds,\ t\in[0,1]$

$$|y(t)|\le|\varphi(t)|+\int_0^1 k(t,s)(a(s)+bx_0(s))ds=x_0(t) \quad \text{a.e. on } [0,1] ;$$

and so $y(t)=0$. This means that $x_0=\theta_{L^1[0,1]}$ solves our equation (1). Now, we assume $x_0\ne\theta_{L^1[0,1]}$ and we consider the following subset of $L^1[0,1]$

$$Q=\{y:\ y\in L^1[0,1],\ |y(t)|\le x_0(t) \text{ a.e.}\}$$

It is clear that Q is nonempty, bounded, closed and convex in $L^1[0,1]$. Define an operator $H:L^1[0,1]\longrightarrow L^1[0,1]$ by putting

$$Hx(t)=\varphi(t)+\int_0^1 k(t,s)f(s,x(s))ds .$$

By virtue of our assumptions H is continuous. We shall prove that i) $H(Q)\subset Q$, ii) $H(Q)$ is relatively compact, so that we are allowed to use Schauder fixed point Theorem to conclude our proof. We start by proving i). Let $x\in Q$ and majorize Hx as follows

$$|Hx(t)|\le|\varphi(t)| + \int_0^1 k(t,s)|f(s,x(s))|ds \le |\varphi(t)| +$$

$$+\int_0^1 k(t,s)(a(s)+b|x(s)|)ds \le |\varphi(t)|+$$

$$+\int_0^1 k(t,s)(a(s)+bx_0(s))ds=x_0(t)$$

by virtue of the Lemma. Now, we show ii) Given $n\in N$, Lusin Theorem and Scorza-Dragoni Theorem alike allow us to find a closed set $A_n\subset[0,1]$, $m(A_n^c)< \frac{1}{n}$ such that $\varphi|_{A_n}$, $k|_{A_n\times[0,1]}$ are uniformly continuous. Now, let (y_k) be a sequence in Q; for t', $t''\in A_n$ we get

$$|Hy_k(t')-Hy_k(t'')|\le|\varphi(t')-\varphi(t'')| +$$

$$+\int_0^1 |k(t',s)-k(t'',s)|(a(s)+bx_0(s))ds.$$

This means that the sequence (Hy_k) is a sequence of equicontinuous functions on A_n, by virtue of the uniform continuity of φ on A_n and k on $A_n\times[0,1]$; being the same sequence equibounded on A_n (easy), we can use Ascoli-Arzelà Theorem ([6]) to prove that (Hy_k) is a relatively compact subset of $C^0(A_n)$; and this can be done for each $n\in N$. We can conclude that there is a suitable subsequence $(y_{k(h)})$ of (y_k) such that $(Hy_{k(h)})$ is a Cauchy sequence in each $C^0(A_n)$, $n\in N$. Now, given $\sigma>0$, let $\rho>0$ be such that $m(A)<\rho$ implies $\int_A x_0(s)ds< \frac{\sigma}{4}$. Choose $\bar{n}\in N$ so that $m(A^c_{\bar{n}})<\rho$ and calculate as follows

$$\int_0^1 |Hy_{k(h')}(t)-Hy_{k(h'')}(t)|dt = \int_{A^c_{\bar{n}}} |Hy_{k(h')}(t)-Hy_{k(h'')}(t)|dt+$$

$$+\int_{A_{\bar{n}}} |Hy_{k(h')}(t)-Hy_{k(h'')}(t)|dt \le$$

$$\le \frac{\sigma}{2} + \int_{A_{\bar{n}}} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\, dt \le \frac{\sigma}{2} +$$

$$+ \|Hy_{k(h')} - Hy_{k(h'')}\|_{C^0(A_{\bar{n}})}$$

Since for h′, h″ sufficiently large the last norm can be made smaller than $\sigma/2$, we obtain the following limit relation

$$\lim_{h',h''\to\infty} \|Hy_{k(h')} - Hy_{k(h'')}\|_{L^1[0,1]} = 0.$$

that concludes the proof. We are done.

Remark 1. With the same proof we can prove the following result valid in the case of $L^p[0,1]$, $1<p<\infty$.

THEOREM 1 (case of $L^p[0,1]$, $1<p<\infty$). *Let the following hypotheses be verified*

(h_1) $\varphi\in L^p(I)$

(h_2) $f:I\times R \rightarrow R$ *verifies Caratheodory hypotheses and there are* $a\in L^p(I)$ *and* $b\geq 0$ *such that*

$$|f(t,x)|\leq a(t)+b|x| \qquad \textit{for a.a. } t\in[0,1],\ x\in R$$

(h_3) $k:I\times I \rightarrow R_+$ *verifies Caratheodory hypotheses and the operator* K *maps* $L^p[0,1]$ *into itself (continuously)*

(h_4) $b\|K\|<1$.

Then the equation (1) *has at least a solution* $x\in L^p[0,1]$.

Remark 2. The proof of our Theorem 1 can be even adapted to the case of arbitrary finite dimensional Banch spaces. So we have the following result.

THEOREM 1'. *Let the following hypotheses be verified*

(h_1) $\varphi\in L^1(I,R^n)$, *with* I *a closed, bounded subset of some* R^m

(h_2) $f:\ I\times R^n \rightarrow R^p$ *verifies Caratheodory hypotheses and there are* $a\in L^1(I)$, $b\geq 0$ *such that*

$$\|f(t,x)\|\leq a(t)+b\|x\|$$

(h_3) $k:\ I\times I \rightarrow L(R^p,R^n)$ (*where* $L(R^p,R^n)$ *denotes the spaces of all linear, bounded operators from* R^p *into* R^n) *verifies Caratheodory hypotheses, is such that the operator*

$$Kx(t) = \int_I K(t,s)x(s)ds$$

maps $L^1(I,R^p)$ *into* $L^1(I,R^n)$ *continuously and moreover* $\|k(\cdot,\cdot)\|:\ I\times I \rightarrow R_+$ *verifies* (h_3) *as in the Theorem* 1

(h_4) $b\|K\|<1$

Then the equation

$$x(t)=\varphi(t)+\int_I k(t,s)f(s,x(s))ds$$

admits a solution $x\in L^1(I,R^n)$.

We have just to observe that the only essential change is the following: instead of Scorza Dragoni Theorem we have to use its generalization due to Ricceri and Villani (see [10]).

As remarked at the beginning, in Theorem 1 we were forced to assume $k(t,s)\geq 0$ for a.a.t, $s\in[0,1]$. In the following result we eliminate this requirement, but only after considering stronger assumptions concerning φ, k, f; this because we look-for solutions of (1) in a different kind of subsets of $L^1[0,1]$.

THEOREM 2. *Let us assume there is* p, $1<p<\infty$, *such that*

(a_1) $\varphi\in L^p[0,1]$

(a_2) $f:[0,1]\times R \to R$ *verifies Caratheodory hypotheses and moreover there exist* $a\in L^p[0,1]$ *and* $b\geq 0$ *such that*

$$|f(t,x)|\leq a(t)+b|x| \qquad \text{for a.a.} t\in[0,1] \text{ and all } x\in R$$

(a_3) $k:[0,1]\times[0,1] \to R$ *verifies Caratheodory hypotheses and it is such that the operator* K *maps* $L^p[0,1]$ *into itself and* $L^1[0,1]$ *into itself*

(a_4) $b\|K\|_p<1$, *where* $\|K\|_p$ *denotes the norm of* K *as an operator from* $L^p[0,1]$ *into itself.*

Then the equation (1) *has a solution in* $L^1[0,1]$.

Proof. Let us consider the following subset Q of $L^1[0,1]$

$$Q = \{x: x\in L^p[0,1],\ \|x\|_p\leq r\}$$

where $r=(\|\varphi\|_p+\|K\|_p\|a\|_p)/(1-b\|K\|_p)$. Q is convex and weakly compact in $L^p[0,1]$ and so it is bounded, closed, convex and uniformly integrable in $L^1[0,1]$ (i.e. $\lim_{m(A)\to 0} \sup_{x\in Q} \int_A |x(t)|dt=0$). We consider the operator H we defined in Theorem 1. H maps $L^p[0,1]$ into itself continuously and $L^1[0,1]$ into itself continuously, thanks to our assumptions. For $x\in Q$, we have

$$\|Hx\|_p = \left(\int_0^1 |Hx(t)|^p dt\right)^{1/p} \leq \|\varphi\|_p +$$

$$+\left(\int_0^1\left|\int_0^1 k(t,s)f(s,x(s))ds\right|^p dt\right)^{1/p}\le$$

$$\le \|\varphi\|_p + \|K\|_p\ \|f(s,x(s))\|_p \le \|\varphi\|_p + \|K\|_p\left(\|a\|_p + b\|x\|_p\right)\le r$$

and so $H(Q)\subset Q$. As in Theorem 1 we can show that $H(Q)$ is relatively compact in $L^1[0,1]$ and so an easy application of Schauder fixed point Theorem concludes our proof. We are done.

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