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On the Banach Spaces with the Property (V^*) of Pelczynski. - II (*)(**).

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Summary. – *We present a converse of a result contained in our paper «On the Banach spaces with the property* (V^*) *of Pelczynski» so obtaining a characterization of that family of Banach spaces. Hence some extensions of other results from that note are presented. At the end we study property* (V^*) *in spaces of operators with compact range.*

1. – Introduction.

Let E be a Banach space and X be a bounded set in E. We say that X is a (V^*) set if for any weakly unconditionally converging series $\sum x_n^*$ in E^* one has

$$\lim_n \sup_X |x_n^*(x)| = 0 .$$

E is said to have property (V^*) if any its (V^*) subset is relatively weakly compact.

Recently, some papers have been devoted to the investigation of this class of Banach spaces ([1], [5], [6], [9], [16]). Main purpose of the present note is to continue the study started in [5] improving some results from that note as well as presenting some new facts concerning property (V^*).

2. – Results about property (V^*).

First of all, we recall the following Proposition from [5]: «Assume E has the property (V^*). Then any conjugate unconditionally converging operator from E^* into F^*, F an arbitrary Banach space, is weakly compact». Now, we present a converse of this

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result, so obtaining a characterization of property (V^*) similar to that one obtained for property (V) by A. PELCZYNSKI in [14], when introducing both properties.

THEOREM 1. – *Assume any conjugate unconditionally converging operator from E^* into F^*, F a Banach space, is weakly compact. Then E has property (V^*).*

PROOF. – Let K be a (V^*) set; hence K is bounded and any its subset is again a (V^*) set. We consider an arbitrary sequence (x_n) in K and we show it is relatively weakly compact. Define T from E^* into l^∞ by $T(x^*) = (x^*(x_n))$. First of all we prove T is a conjugate operator. Assume $x_\alpha^* \to \theta$ weak*. If $y \in l^1$, we have to show that for each $\eta > 0$ there is α_0 such that for any $\alpha \geqslant \alpha_0$ we have $|\langle y, T(x_\alpha^*)\rangle| < \eta$. If $y \in l^1$, it has an expansion of the following type: $y = \sum_{i=1}^{\infty} b_i e_i$ with $\sum_{i=1}^{\infty} |b_i| < +\infty$. Hence, if $y_n = \sum_{i=1}^{n} b_i e_i$ we have $\|y - y_n\| \to 0$. Take n' such that $\|y - y_{n'}\| < \eta/(2\|T\| \sup_\alpha \|x_\alpha^*\|)$. Then we have

$$|\langle y, T(x_\alpha^*)\rangle| \leqslant \|y - y_{n'}\| \|T\| \sup_\alpha \|x_\alpha^*\| + |\langle y_{n'}, T(x_\alpha^*)\rangle| \leqslant \eta/2 + \left| \sum_{i=1}^{n'} b_i x_\alpha^*(x_i) \right| \leqslant \eta/2 + \sum_{i=1}^{n'} |b_i| \, |x_\alpha^*(x_i)| .$$

Now, since $x_\alpha^* \to \theta$ weak*, given $\eta \Big/ \left(2 \sum_{i=1}^{\infty} |b_i|\right)$ there is α_0 such that, for $\alpha \geqslant \alpha_0$, the following is true

$$|x_\alpha^*(x_i)| < \eta \Big/ \left(2 \sum_{i=1}^{\infty} |b_i|\right),$$

for each $i = 1, 2, \ldots, n'$. Hence for $\alpha > \alpha_0$ we get $|\langle y, T(x_\alpha^*)\rangle| < \eta$, i.e. $T(x_\alpha^*) \to \theta$ weak*. We are done. The second step is to show that T is unconditionally converging. Let $\sum x_n^*$ be weakly unconditionally converging, but $\sum T(x_n^*)$ is not unconditionally converging; hence there is a permutation $h(n)$ on N such that $\sum T(x_{h(n)}^*)$ doesn't converge. There exist $\eta > 0$ and $p_1 < q_1 < p_2 < q_2 < \ldots < p_n < q_n < \ldots$ such that

$$\left\| \sum_{i=p_n}^{q_n} T(x_{h(i)}^*) \right\| > \eta . \tag{1}$$

Consider, now, $y_n^* = \sum_{i=p_n}^{q_n} x_{h(i)}^*$ and observe that $\sum y_n^*$ is weakly unconditionally converging and so $\sup_k |y_n^*(x_k)| \to 0$; but this is against (1). We are done. Hence, the operator $S: l^1 \to E$ such that $S^* = T$ is weakly compact, by our assumption. It is now very easy to see that $\{S(e_n): n \in N\}$ contains (x_n). This concludes the proof.

In order to go ahead with our study we need the definition of (RDP) property: E is said to possess the (RDP) property if any Dunford-Pettis operator from E into an arbitrary Banach space F is weakly compact, and (RDP*) property: E is said to possess the (RDP*) property if any its Dunford-Pettis subset is relatively weakly compact.

REMARK 1. – As above (and in [5]) we are able to give the following characterization of Banach spaces with the (RDP*) property:

E has the (RDP*) *property iff each conjugate Dunford-Pettis operator from E^* into F^*, F an arbitrary Banach space, is weakly compact.*

In the previous paper [5] we also obtained the following result: «Let E be a Banach lattice. E has the (RDP) property iff E^* has the property (V^*)». Here we observe that the following extension is true

THEOREM 2. – *Let E be a Banach space complemented in a Banach lattice F. Then the following are equivalent:*

(*a*) *E has the* (RDP) *property,*

(*b*) *E doesn't contain complemented copies of l^1,*

(*c*) *E^* has the* (RDP*)*property,*

(*d*) *E^* doesn't contain copies of c_0,*

(*e*) *E^* is weakly sequentially complete,*

(*f*) *E^* has property* (V^*).

PROOF. – That (*a*) implies (*b*) and (*c*) is true in general Banach spaces. Assume (*c*) is true; it is well known ([5]) that the (RDP*) property is inherited by closed subspaces; so, if c_0 lived in E^* it should enjoy and the (RDP) property and the (RDP*) property, which is not possible for a nonreflexive Banach space. Hence (*d*) is true. Assume, now, (*d*). Since E is complemented in F, E^* is complemented in F^*. Hence, by a result in [11] E^* is contained in a Banach lattice Z not containing a copy of c_0, i.e. a weakly sequentially complete Banach lattice; hence (*e*) is true. That (*e*) gives (*f*) can be seen as the implication $(d) \Rightarrow (e)$, because (*d*), (*e*) and (*f*) are equivalent in Banach lattices (see [5]), $(f) \Rightarrow (c)$ follows from the definitions. Since (*f*) implies (*d*), (*f*) implies (*b*), too. In order to conclude the proof, we need only to prove that (*b*) gives (*a*). Assume (*b*). Theorem 1.2 in [12] gives that E is complemented in a Banach lattice Z not containing complemented copies of l^1 Theorem 2.1 in [13] gives that (Z and hence) E verifies (*a*). The proof is complete.

The above Theorem 2 also proves that (*i*) of Remark 2 in [5] can be reversed in some case, even if this is not longer true in general Banach spaces ([5]). We also underline that the examples considered in the paper [5] (i.e. the James space J and the Bourgain-Delbaen space X with Schur property) prove that in general Banach spaces there is no relationship between the (RDP) property in a Banach space E and property (V^*) in its dual E^*. Indeed, J has the (RDP) property, but J^* doesn't have property (V^*); X doesn't possess the (RDP) property, but its dual space has property (V^*) because it is a weakly sequentially complete Banach lattice (see Theorem 1.7 of [5]). Some corollaries of Theorem 2 now follow.

COROLLARY 3. – *Let E be a Banach space complemented in a Banach lattice. Then if E^{**} has the* (RDP) *property, E itself has that property.*

PROOF. – Under our assumptions, E^{***} has the (RDP*) property, hence it cannot contain a copy of c_0 as well as E^* does. Theorem 2 now concludes the proof.

The above corollary has some interest since in general Banach spaces the (RDP) property is not inherited by closed subspaces; the already quoted Bourgain-Delbaen space X doesn't have the (RDP) property, whereas its bidual X^{**} is an injective Banach space; so it is complemented in a suitable $C(K)$ and it has the (RDP) property.

COROLLARY 4. – *Let E be a Banach space complemented in a Banach lattice F. If K is a compact Hausdorff space, then $C(K, E)$ has the* (RDP) *property iff E has.*

PROOF. – $C(K, E)$ is complemented in the Banach lattice $C(K, F)$. By virtue of a result of Talagrand ([18]) $C(K, E)^*$ is weakly sequentially complete iff E^* is. Theorem 2, now, works.

The proof of Theorem 2 again is useful to get the following slight extension of recent results by Bombal ([1]) and Leung ([9]), who showed the equivalences $(b') \Leftrightarrow (c') \Leftrightarrow (d')$.

PROPOSITION 5. – *Let E be a Banach space complemented in a Banach lattice. Then the following are equivalent*

(a') *E has the* (RDP*) *property,*

(b') *E doesn't contain copies of c_0,*

(c') *E is weakly sequentially complete,*

(d') *E has the property* (V^*).

Theorem 2 also admits the following partial improvement. that can be obtained using Proposition 5.

THEOREM 6. – *Assume E is a Banach space such that E^* is complemented in a Banach lattice. If (a)-(f) are like in Theorem* 2, *then* $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$.

Observe that any Banach space with G.L.l.u.st. ([7]) verifies the hypothesis of Theorem 6. But we cannot have $(a) \Leftrightarrow (b)$ since again the Bourgain-Delbaen space is a counterexample. If we consider Banach spaces with l.u.st. ([3]) the following result is true (in it L^1 will denote the usual Lebesgue-Bochner function space on a finite measure space).

THEOREM 7. – *Let E be a Banach space with l.u.st.. Assume E doesn't contain l_n^∞ uniformly for all $n \in N$. Then E (as well as $L^1(\mu, E)$) has the property* (V^*).

PROOF. – It is known that E is contained in a Banach lattice F which is finitely represented in E. So F cannot contain copies of c_0. Theorem 1.7 of [5] gives that (F and hence) E has the property (V^*). Moreover, $L^1(\mu, E)$ is a closed subspace of the Banach lattice $L^1(\mu, F)$ that enjoys the property (V^*) ([5]).

3. – The property (V^*) in spaces of compact operators.

The last part of the note is devoted to the study of the property (V^*) in the space $K_{w^*}(E^*, F)$ of weak*-weakly continuous compact operators from E^* into F, E, F two arbitrary Banach spaces.

THEOREM 8. – *$K_{w^*}(E^*, F)$ has the property (V^*) iff it is weakly sequentially complete and E and F have the same property.*

PROOF. – One implication is trivial. Hence assume $K_{w^*}(E^*, F)$ is weakly sequentially complete and E and F have property (V^*). Take a (V^*) set M in $K_{w^*}(E^*, F)$; we may assume $M = (h_n)$. We want to show that a suitable subsequence of (h_n) will converge weakly. Observe that $\overline{\text{span}}\,\{h_n(x^*) \colon x^* \in E^*,\ n \in N\}$ is a separable closed subspace of F; so we may suppose (and we do) that F is separable. Now, there is a countable set $Y \subset F^*$ such that span(Y) in w^*-dense in F^*. For each $y^* \in Y$ we get a (V^*) set, $\{h_n^*(y^*) \colon n \in N\}$ in E, that has the property (V^*). Hence there is a subsequence, say $(h_{k(n)})$, so that $(h_{k(n)}^*(y^*))$ converges weakly; since Y is countable, we may assume that $(h_{k(n)}^*(y^*))$ is weakly Cauchy for each $y^* \in Y$. Taking $x^* \in E^*$ we consider, now, $(h_{k(n)}(x^*))$ that is a (V^*) set in F and hence is relatively weakly compact. We shall show that any weakly converging subsequence has to converge to the same element; this will imply that, actually, the sequence $(h_{k(n)}(x^*))$ converges weakly in F. Assume that y' is the limit of a suitable subsequence $(h_{p(n)}(x^*))$; for $y^* \in Y$ we have

$$y^*(y') = \lim_n y^*(h_{p(n)}(x^*)) = \lim_n h_{p(n)}^*(y^*)(x^*) = \lim_n h_n^*(y^*)(x^*) = \lim_n y^*(h_n(x^*)) .$$

If y'' is another (sequential) cluster point of $(h_{k(n)}^*(x^*))$, as above, we get

$$y^*(y'') = \lim_n y^*(h_n(x^*)) .$$

From the above equalities we have

$$y^*(y') = y^*(y'') \qquad \text{for all } y^* \in Y .$$

But Y is w^*-dense in F^* and so $y' = y''$. Hence $(h_n(x^*))$ is weakly converging in F, for all $x^* \in E^*$. If we take $y^* \in F^*, \|y^*\| \le 1$, the real sequence is thus converging. Now, observe that (see [15]) ext (dual unit ball of $K_{w^*}(E^*, F)$) = ext (dual unit ball of E) $\otimes$ ext (dual unit ball of F). By the Rainwater Theorem ([2],

p. 156) (h_n) is a weak Cauchy sequence. The weak completeness of the considered space $K_{w^*}(E^*,F)$ concludes the proof.

REMARK 2. – Observe that in [1] the weak property (V^*) was defined: a Banach space has the weak property (V^*) if any its (V^*) set is conditionally weakly compact. Actually, the proof of Theorem 8 shows that if E has the property (V^*) and F the weak property (V^*), then $K_{w^*}(E^*,F)$ has the weak property (V^*).

It is known that if E and F are weakly sequentially complete and E (or F) has unconditional basis, then $K_{w^*}(E^*,F)$ is weakly sequentially complete ([10]), provided any operator from E^* into F is compact. Furthermore, note that the same is true if F has just the Schur property (i.e. weak Cauchy sequences are norm converging). Here is a proof: let (h_n) be a weak Cauchy sequence in $K_{w^*}(E^*,F)$. Since E and F are weakly sequentially complete, we can define u: $E^* \to F$ by $u(x^*) = w - \lim_n h_n(x^*)$ and also v: $F^* \to E$ by $v(y^*) = w - \lim_n h_n^*(y^*)$. It is easy to see that $u^* = v$ a fact that implies that u is weak*-weakly continuous. Since F has the Schur property, u is compact, too. Since for $x^* \otimes y^* \in E^* \otimes F^*$ we have

$$\lim_n h_n(x^* \otimes y^*) = u(x^* \otimes y^*);$$

hence $h_n \to u$ weakly.

Observe that l^1 is a Banach space with and Schur property and property (V^*); but also the spaces constructed by HAGLER ([8]) and TALAGRAND ([17]) have duals enjoying both of these properties without being isomorphic to l^1.

Let (S, Σ, μ) be a finite measure space and E be a Banach space. We consider the normed space $P_c(\mu, E)$ of all (classes of weakly equivalent) weakly measurable Pettis integrable functions having an indefinite integral with compact range, equipped with the norm

$$\|f\| = \sup\left\{ \int_S |x^*f(s)|\, d\mu \colon x^* \in E^*, \|x^*\| \leq 1 \right\}.$$

It is well known that it is not complete. We have the following result concerning the completion $\overline{P_c(\mu, E)}$ of $P_c(\mu, E)$.

THEOREM 9. – *The completion* $\overline{P_c(\mu, E)}$ *of* $P_c(\mu, E)$ *is isometrically isomorphic to a closed subspace of* $K_{w^*}(E^*, L^1(\mu))$.

PROOF. – It will suffice to show that $P_c(\mu, E)$ is isometrically isomorphic to a subspace of $K_{w^*}(E^*, L^1(\mu))$. For any $f \in P_c(\mu, E)$ define an operator T_f: $E^* \to L^1(\mu)$ by $T_f(x^*) = x^*f$. It is well-known that T_f is weak*-norm continuous ([4]) and hence it is an element of $K_{w^*}(E^*, L^1(\mu))$. It is quite simple to show that the mapping $f \to T_f$ is one-to-one and $\|f\| = \|T_f\|$. The proof is over.

Now, we can apply Theorem 8 to $\overline{P_c(\mu, E)}$ by using Theorem 9.

THEOREM 9. – *$\overline{P_c(\mu, E)}$ has the property (V^*) iff it is weakly sequentially complete and E has the same property. In particular this happens if either*

(a) E has an unconditional basis, property (V^*) and any operator from E^* into $L^1(\mu)$ is compact,

(b) E has the Schur property and property (V^*).

Notes added.

A. PELCZYNSKI ([14]) showed that if E has the property (V), then E^* has property (V^*) and put the question of whether the converse implication is true. In [5] we proved that this question has a negative answer, even in Banach lattices. Here we present a result showing that question can be answered positively in the class of closed subspaces of a Banach lattice with order continuous norm; actually we shall prove more, since the result we state involves other isomorphic properties. In the paper «A. GROTHENDIECK: *Sur les applications linéaires faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math., 5 (1953), pp. 129-173» the author introduced the *(D) property*: a Banach space E has the (D) property if each weakly completely continuous operator defined on E is weakly compact. We have the following result which involves a lot of isomorphic properties and establishes relationships among them in a special class of Banach spaces. We omit the proof, because it is a quite easy consequence of the definitions and a well known result by L. Tzafriri, stating that *a closed subspace of an order continuous Banach lattice contains a complemented copy of l^1, provided it contains a copy of l^1* (see L. TZAFRIRI: *Reflexivity in Banach lattices and their subspaces*, J. Funct. Analysis, **10** (1972), pp. 1-18).

THEOREM. – *Let E be a closed subspace of an order continuous Banach lattice. Then the following facts are equivalent:*

(i) *E^* has the property (V^*),*

(ii) *E^* is weakly sequentially complete,*

(iii) *E^* has the* (RDP)* *property,*

(iv) *E^* doesn't contain copies of c_0,*

(v) *E doesn't contain copies of l^1,*

(vi) *E has the (D) property,*

(vii) *E has the* (RDP) *property,*

(viii) *E has the property (V).*

The following corollary of the above theorem seems worth being pointed out

COROLLARY. – *Let E be a closed subspace of an order continuous Banach lattice. If E^{**} doesn't contain a complemented copy of l^1, then E has the property (V).*

PROOF. – If E^{**} is like above, then (iv) of Theorem is true, because c_0 doesn't embed into E^{***}. The Theorem concludes.

We note that if E^{**} had property (V), it couldn't contain complemented copies of l^1; hence, E would inherit that property from E^{**}. This fact is not true for general Banach spaces, because of the separable $\mathcal{L}_\infty$-space constructed in «J. BOURGAIN - F. DELBAEN: *A special class of* $\mathcal{L}_\infty$*-spaces*, Acta Math., **145** (1980), pp. 155-176»; indeed, such a space has the Schur property and hence cannot possess the property (V), but its bidual is complemented in a $C(K)$ space and so it has the property (V).

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