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PRECOMPACTNESS IN THE SPACE OF PETTIS INTEGRABLE FUNCTIONS

G. EMMANUELE (Catania)[1]

Let (S, Σ, μ) be a finite measure space and X a Banach space. We consider the normed space $\mathsf{P}_c(\mu, X)$ of all (μ-) Pettis integrable functions, with values into X, having an indefinite integral with compact range; the norm in $\mathsf{P}_c(\mu, X)$ is, as usual,

$$\|f\| = \sup\left\{ \int_S |x^* f(s)| d\mu : x^* \in X^*, \|x^*\| \leqq 1 \right\}.$$

The purpose of this note is to extend a characterization of precompact subsets obtained by Brooks and Dinculeanu in [1] for strongly measurable functions and by Graves and Ruess in [2] for subsets of bounded weakly measurable functions defined on perfect measure spaces. Those results were obtained as consequences of other theorems about spaces of unconditionally converging series (cf. [1]) and about spaces of compact range vector measures (cf. [2]); our approach is direct and this makes it possible to consider the most general case.

Before giving our result we need to state some terminology. If $\pi = (A_i)_{i \in I}$ is a finite partition of S, we define the conditional expectation $E(\pi, \mu)f$ of f by

$$E(\pi, \mu)f = \sum_{i \in I} [\mu(A_i)]^{-1} \left(\int_{A_i} f(s) d\mu \right) \Phi_{A_i}.$$

It is known (cf. [3]) that the family of finite partitions is directed by refinement, that $\|E(\pi, \mu)f\| \leqq \|f\|$ and that $\lim_\pi \|E(\pi, \mu)f - f\| = 0$ for all $f \in \mathsf{P}_c(\mu, X)$.

Our result is contained in the following theorem.

THEOREM 1. *Let H be a bounded subset of $\mathsf{P}_c(\mu, X)$. The following facts are equivalent:*

(a) *H is precompact,*

(b) (i) $\left\{ \int_A f(s) d\mu : f \in H \right\}$ *is relatively compact in X for all $A \in \Sigma$,*

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(ii) $\lim_\pi E(\pi,\mu)f = f$ *uniformly on* $f \in H$.

PROOF. (a) $\Rightarrow$ (b, i) is a direct consequence of the fact that the operator $f \to \int_A f(s)d\mu$ is linear and continuous.

(a) $\Rightarrow$ (b, ii). Observe that H is totally bounded and so, given $\varepsilon > 0$, there are $f_1, f_2, \dots, f_n \in \mathsf{P}_c(\mu, X)$ such that any $f \in H$ is at a distance less than $\varepsilon/3$ from some f_i. Hence we get

$$\|E(\pi,\mu)f - f\| \leqq \|E(\pi,\mu)f - E(\pi,\mu)f_i\| + \|E(\pi,\mu)f_i - f_i\| +$$

$$+\|f_i - f\| \leqq 2\|f_i - f\| + \|E(\pi,\mu)f_i - f_i\| \leqq (2/3)\varepsilon + \|E(\pi,\mu)f_i - f_i\|.$$

Since as already recalled $\lim_\pi \|E(\pi,\mu)f - f\| = 0$ for all f, there is a π' such that, if $\pi > \pi'$, then $\|E(\pi,\mu)f_i - f_i\| \leqq \varepsilon/3$ for $i = 1, 2, \dots, n$; the above inequalities conclude the proof.

(b) $\Rightarrow$ (a). Consider a sequence in H and observe that, for $n, m \in N$,

$$\|f_n - f_m\| \leqq \|f_n - E(\pi,\mu)f_n\| + \|E(\pi,\mu)f_n - E(\pi,\mu)f_m\| +$$

$$+\|E(\pi,\mu)f_m - f_m\|.$$

Using (ii) we can find a π_k, $k \in N$, such that

$$\|E(\pi_k,\mu)f - f\| \leqq 1/k \quad \text{uniformly on} \quad f \in H.$$

Moreover, by virtue of (i) we can assume (otherwise we pass to a subsequence) that $(E(\pi_k,\mu)f_n)$ is a Cauchy sequence. The inequalities considered above allow us to conclude our proof.

REMARK 1. In a sense the result above is the best possible, because if H is a set of Pettis integrable functions (it does not matter how the range of the indefinite integral is) for which Theorem 1 is true, then $f \in H$ must be in $\mathsf{P}_c(\mu, X)$ because of a result of Musial [3].

If we consider only sequences of partitions we have the following result:

THEOREM 2. *Let H be a bounded subset of $\mathsf{P}_c(\mu, X)$. The following facts are equivalent:*

(a) *H is precompact,*

(b) (i) *see Theorem* 1,

(ii) *for any sequence $(f_k) \subset H$ there is a sequence (π_n) of finite partitions, cofinal to the net (π), such that*

$$\lim_n \|E(\pi_n,\mu)f_k - f_k\| = 0 \quad \textit{uniformly on} \quad k \in N.$$

PROOF. That (b) implies (a) is similar to the proof of the same implication in Theorem 1. So we have just to show that (a) implies (b). Of course (b, i) is clear. Now, consider a sequence $(f_k) \subset H$; it is well known (cf. [4])

that there is a sequence (π_n) of finite partitions cofinal to the net (π) so that, for all $k \in N$, one has

$$\lim_n \|E(\pi_n, \mu)f_k - f_k\| = 0.$$

Using, as in Theorem 1, the total boundedness of (f_k) we can finish our proof.

References

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DEPARTMENT OF MATEHEMATICS
UNIVERSITY OF CATANIA
VIALE DORIA 6
95125 CATANIA
ITALY