Differential Gröbner Bases, Resultants, etc.

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In Pina’s Memoriam
Contents

1. Differential Gröbner Bases
2. Differential Resultants
3. Extended Characteristic Sets
4. Involutive Divisions
5. Conclusion
Algebraic Gröbner Bases

Definition: (Buchberger’76)

Given a field \( K \), an ideal \( I \subset \mathbb{R} := K[x_1, \ldots, x_n] \) and an admissible monomial (power product) order \( \succ \), a finite set \( G \subset I \) is (algebraic) Gröbner basis of \( I \) if

\[
(\forall f \in I) \ (\exists g \in G) : \text{lm}(g) | \text{lm}(f)
\]

Below we shall comment on some of the (open that time) problems formulated by G.Carrà Ferro in:

Differential Gröbner Bases

Notations:
- $\mathcal{K}$ - differential field with the set of derivations $\Delta := \{\delta_1, \cdots, \delta_m\}$
- $\Theta$ - the monoid of derivation operators generated by $\Delta$
- $\mathbb{R} := \mathcal{K}\{y_1, \ldots, y_p\}$ - the differential polynomial ring with (differential) indeterminates $Z_1, \ldots, Z_p$
- $\prod_{i_1}(\vartheta_{i_1}Z_1)^{i_1} \cdots \prod_{i_k}(\vartheta_{i_k}Z_p)^{i_k}$ ($\vartheta_{i_j} \in \Theta$) - differential monomial
- $\succ$ - ranking and $>$ - differential term order compatible with $\succ$

\[ \theta Z_i \succ \vartheta Z_j \implies \theta Z_i > \vartheta Z_j \quad \forall \vartheta \in \Theta, \forall i, j \in \{1, \ldots, p\} \]

Given a finitely generated differential ideal $\mathcal{I} := \langle f_1, \ldots, f_r \rangle \subset \mathbb{R}$ and a term order $>$ compatible with ranking, a finite set $G$ s.t. $\mathcal{I} = \langle G \rangle$ and

\[ (\forall f \in \mathcal{I}) \ (\exists g \in G, \theta \in \Theta) : \text{lm}(\theta g) | \text{lm}(f) \]

is called a differential Gröbner basis of $\mathcal{I}$.
If $|G| = \infty$, $G$ is called a standard basis (Ollivier’90).
Citations

Representation for the radical of a finitely generated differential ideal - all 9 versions

Page 1. Representation for the radical of a finitely generated differential ideal* F. Boulier UFL – Université Libre de Bruxelles, Belgium
CEDEX boulier(at)ulb.be and F. Ollivier UCL – Université Libre de Bruxelles, Belgium
Cited by: 90 - Related Articles - Web Search

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In this paper we consider an algorithmic technique more general than that proposed by Zharkov and Blinkov for the involutive analysis of polynomial ideals. It is based on a new concept of involutive monomial division which ... Cited by: 130 - Related Articles - View as HTML - Web Search

Factorization-free Decomposition Algorithms in Differential Algebra - all 9 versions
E. Hubert - Journal of Symbolic Computation, 2000 - Elsevier

This paper makes a contribution to differential elimination and more precisely to the problem of computing a representation of the radical differential ideal generated by a system of differential equations (ordinary or partial). The ... Cited by: 89 - Related Articles - Web Search

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  - Differential-algebraic equations (DAE) and partial differential-algebraic equations (PDAE) are systems of ordinary equations and DAEs with constraints. They occur frequently in such applications as constrained multi-body ...
  - Cited by 34 - Related Articles - Web Search

- Algorithms for Symmetric Differential Systems - all 6 versions
  - E. L. Mansfield - Foundations of Computational Mathematics, 2001 - Springer
  - Abstract: Over-determined systems of partial differential equations may be studied using differential-algebraic algorithms, as a great deal of information about the solution set of the system may be obtained from the output ...
  - Cited by 31 - Related Articles - Web Search

- The Differential Ideal [P] - all 3 versions
  - Cited by 20 - Related Articles - Web Search
A point symmetry group of a differential equation which cannot be found using infinitesimal methods
Page 1. A Point Symmetry Group of a Differential Equation which cannot be found using infinitesimal methods. GJ Reid, DT Walch, AD Wittkopf - Modern Group Analysis: Advanced Analytical and Computational Tools, 1999 - cecm.sfu.ca
Cited by 3 - Related Articles - Web Search

Resolvent Representation for Regular Differential Ideals - all 4 versions
Given a system of polynomial differential equations there are several known algorithms to decompose its solution into the union of the non singular solution sets of differential systems of a specific form on which ...
Cited by 13 - Related Articles - Web Search

Parameter identifiability of nonlinear systems: the role of initial conditions - all 4 versions
Filis, S., Audoly, L. - Automatica, 2003 - Elsevier
Identifiability is a fundamental prerequisite for model identification; it concerns uniqueness of the model parameters determined from the input-output data, under ideal conditions of noise-free observations and error-free ...
Cited by 16 - Related Articles - Web Search

Geometry and Structure of Lie Pseudogroups from Infinitesimal Defining Systems - all 8 versions
Lisle, G. - J. Symbolic Comput., 1996 - Elsevier
Cited by 12 - Related Articles - Web Search

Reduced Gröbner Bases, Free Difference-Differential Modules and Difference-Differential Dimension - all 8 versions
We define a special type of reduction in a free left module over a ring of difference-differential operators and use the idea of the Gr"obner basis method to develop a technique which allows us to determine the Hilbert ...
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Citations (cont.)

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Global identifiability of nonlinear models of biological systems
TOC View - Biomedical Engineering, IEEE Transactions on, 2001 - ieeexplore.ieee.org
Abstract—A prerequisite for well-posedness of parameter estimation of biological and physiological systems is a priori global identifiability, a property which concerns uniqueness of the solution for the unknown model...
Related Articles - Web Search

Membership problem for differential ideals generated by a composition of polynomials - all 5 versions »
MV Kondratieva, AI Zobnin - Programming and Computer Software, 2006 - Springer
1. INTRODUCTION The problem of whether a differential polynomial belongs to a finitely generated differential ideal is solved at present only in some particular cases. Among them are the following: the case of radical ideals, ...
Related Articles - Web Search - BL Direct

Boris Kuznetsov NOTE ON TWO COMPATIBILITY CRITERIA: JACOBI-MAYER BRACKET VS. DIFFERENTIAL GROEBNER ... - all 4 versions »
JMB VS - lim.kau.ru
In this paper we investigate and compare two recent results on compat- ibility of overdetermined systems of partial differential equations, which we formulate below. For simplicity of exposition we restrict to the case of scalar PDEs, ...
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H Hiptmair, R Hennemeeke, S Landsman, J Winkler - Mathematical and Computer Modelling of Dynamical Systems, 2004 - informaworld.com
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Computer difference-differential Groebner Bases and difference-differential dimension polynomials] - all 2 versions »
A Zhou, A Winkler - ncs.umn.edu/at
Abstract Difference-differential Groebner bases and the algorithms were introduced by M. Zhou and A. Winkler (2002). In this paper we will make further investigations for the key concept of S-polynomials in the algorithm and we ...
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Differential Gröbner Bases, etc.
Open Problems (CF’94)

- Find relations between Gröbner bases and either involutive or formally integrable (differential) systems

Some relations:
Involutive algebraic and linear differential systems are exactly (generally redundant) Gröbner bases. (Zharkov, Blinkov’93; G., Blinkov’96; Apel’98; G.99).

- Find relations between either characteristic set of coherent set for a system and either involutive or formally integrable systems.

Some relations:
Mansfield’s (Mansfield’91) differential Gröbner basis as an extension of characteristic sets for differential ideals.
Formal integrability $\iff$ coherence (Minzlaff’06).
Open Problems (cont.)

- Characterization of differential term orderings, and, in particular, elimination orderings.

Classification of rankings: (Rust, Reid’97).

Differential term orders: (Weispfenning’93; Carrà Ferro, Zit’94; Zobnin’04)

Differential term orders and finiteness criteria for Gröbner bases: (Zobnin’05)
Differential Resultants

The notion of differential resultant for two nonlinear ordinary differential polynomials (Ritt’32) and for a finite number of linear ordinary differential operators (Berkovich,Tsirulik’86) was extended by G.Carrà Ferro to linear PDEs in:


and to nonlinear ODEs:


Multinomial Resultant for ODEs

Definition: (CF’97)

Let \( \{f_1, \ldots, f_{m+1}\} \subset \mathbb{K}\{Z_1, \ldots, Z_m\} \) such that \( \text{ord}(f_i) = n_i \) (max. order of \( \delta \) occuring in \( f_i \)) and \( N := \sum_{i=1}^{m+1} n_i \). Then the differential resultant \( \text{Res}(f_1, \ldots, f_{p+1}) \) is the Macalau algebraic resultant (Macalau’16) of \( (m + 1)N + 1 \) variables

\[ \delta^{N-n_1}(f_1), \ldots, \delta(f_1), \ldots, \delta^{N-n_{m+1}}(f_{m+1}), \ldots, \delta(f_{m+1}), f_{m+1} \]

Open problem (CF’94): Extension of the notion of differential resultant to PDEs.

Solved for linear PDEs (CF’97) and still unsolved for nonlinear PDEs.
Application of Resultants: Example

Differential resultants can be used (CF’94) to:

1. determine of the orders of derivatives that are necessary to eliminate a differential variable;

2. obtain the (compatibility) condition on the coefficients of a system for the existence of its solution.

Example:

\[ f_1 := \dot{x}^2 - x - a, \quad f_2 := a x^2 - 1, \quad f_1, f_2 \in \mathbb{K}\{a\}\{x\}, \quad \dot{x} := \delta(x) \]

The Macaulay resultant easily computed by the Maple package (http://minimair.org/mr.html)

\[ \text{Res}\{\dot{x}^2 - x - a, a x^2 - 1, a x^2 + 2 a x \dot{x}, \}\{x, \dot{x}\}) = \dot{a}^4 - 8 \dot{a}^2 a^4 + 16 a^8 - 16 a^5 \]

is just the differential resultant
Extended Characteristic Sets

In


we shown the following

If one computes an algebraic Gröbner basis of the input set of differential polynomials and then starts the Kolchin-Ritt algorithm (Mansfeld’91) with this basis, then the output characteristic set has rank equal or less than that obtained by the algorithm without the algebraic Gröbner basis computation.

By this reason we called the output of the improved algorithm by extended characteristic set of the input set of polynomials.
Examples

Example 1: \( F := \{ xy^2, yz - 1, xv - \partial z \} \).
Fix the elimination ranking with \( x \prec y \prec z \prec v \). Then the Kolchin-Ritt algorithm outputs \( G := \{ xy^2, yz - 1 \} \). algebraic Gröbner basis for \( F \) we obtain \( \tilde{F} := \{ x, yz - 1, \partial z \} \) and the Kolchin-Ritt algorithm for the input \( \tilde{F} \) gives the output \( \tilde{G} := \{ x, \partial y, yz - 1 \} \) with \( \text{rank}(\tilde{G}) < \text{rank}(G) \).

Example 2: \( F := \{ xy^2, y\partial_1 z - x, \partial_2 z \} \).
Fix the elimination ranking
\[
x \prec y \prec z, \quad \partial_1 x \prec \partial_2 x \prec \partial_1 y \prec \partial_2 y \prec \partial_1 z \prec \partial_2 z
\]
Then the Kolchin-Ritt algorithm for \( F \) outputs \( G := \{ xy^2, y\partial_1 z - x, \partial_2 z \} \). If we aly the algorithm to the Gröbner basis \( \tilde{F} = \{ x^3, x^2 y, xy^2, y\partial_1 z - x, \partial_2 z \} \) of \( F \) we obtain \( \tilde{G} := \{ x^3, x^2 y, \partial_2 z \} \) with \( \text{rank}(\tilde{G}) < \text{rank}(G) \).
Involution Division

Let $\mathbb{M}$ be the set of monomials (power products) in $\mathbb{R} := \mathbb{K}[x_1, \ldots, x_n]$.

**Definition:** (G., Blinkov’98) An involutive separation $\mathcal{L}$ of variables is defined on $\mathbb{M}$ if for any finite set $U \subset \mathbb{M}$ and for any $u \in U$ there is defined a subset $M_{\mathcal{L}}(u, U) \subseteq \{x_1, \ldots, x_n\}$ of variables generating monoid $\mathcal{L}(u, U) \equiv \mathbb{M}_{M(u, U)}$ such that

1. $u, v \in U, \ u\mathcal{L}(u, U) \cap v\mathcal{L}(v, U) \neq \emptyset \iff u \in v\mathcal{L}(v, U) \text{ or } v \in u\mathcal{L}(u, U)$.
2. $v \in U, \ v \in u\mathcal{L}(u, U) \iff \mathcal{L}(v, U) \subseteq \mathcal{L}(u, U)$.
3. $V \subseteq U \implies \mathcal{L}(u, U) \subseteq \mathcal{L}(u, V) \forall u \in V$.

Variables in $M_{\mathcal{L}}(u, U)$ are called ($\mathcal{L}$−)multiplicative for $u$ and those in $NM_{\mathcal{L}}(u, U) := \{x_1, \ldots, x_n\} \setminus M(u, U)$ are ($\mathcal{L}$−)nonmultiplicative for $u$, respectively.

involution separation $\iff$ involutive division

If $w \in u\mathcal{L}(u, U)$ then $u$ is involutive divisor of $w$: $u \mid_{\mathcal{L}} w \implies$ involutive reduction and involutive normal form $NF_{\mathcal{L}}(f, F)$ where $f \in \mathbb{R}$, $F \subset \mathbb{R}$. 

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Involutional Bases

Definition: (G., Blinkov’98; G.’05)

Similarly to the above definition of algebraic Gröbner basis, an (involutive) \( \mathcal{L} \)-basis \( H \) of \( I \subset \mathbb{K}[x_1, \ldots, x_n] \) is defined as

\[
(\forall f \in I) \ (\exists h \in H) : \ \text{lm}(g) \mid_{\mathcal{L}} \text{lm}(f)
\]

Since

\[
u \mid_{\mathcal{L}} v \implies u \mid v
\]

it follows that an involutive basis is a (generally redundant) Gröbner basis.
Janet Division

Definition: (Janet’20; G., Blinkov’98) For each finite monomial set $U \subset M$ and $0 \leq i \leq n$ partition $U$ into groups labeled by $d_0, \ldots, d_i \in \mathbb{N}_{\geq 0}$ ($U = [0]$)

$$[d_0, d_1, \ldots, d_i] := \{u \in U \mid d_0 = 0, d_1 = \deg x_1(u), \ldots, d_i = \deg x_i(u)\}. $$

Variable $x_i$ is $\mathcal{J}$-multiplicative for $u \in U$ if $u \in [d_0, \ldots, d_{i-1}]$ and

$$\deg_i(u) = \max\{\deg_i(v) \mid v \in [d_0, \ldots, d_{i-1}]\}.$$
Characterization of Involutive Divisions

In:


the static properties of involutive divisions, i.e. those satisfying the above axioms (1)-(2), are studied.

In particular,

- Given a finite set $U$, all possible involutive divisions restricted to $U$ are found.
- An algorithm is presented to construct such a restricted set of divisions.
The Main Result

Theorem:

Let $U$ be a set of monomials, $\mathcal{L}$ an involutive division, $u, v \in U$, $C_\mathcal{L}(u, U) := u\mathcal{L}(u, U)$ the $\mathcal{L}$–cone of $u$ and

$$A_{u,v} := \{x_i | \deg x_i(u) < \deg x_i(v)\}, \quad E_{u,v} := \{x_i | \deg x_i(u) = \deg x_i(v)\}$$

Then

1. $NM_\mathcal{L}(u, U) \cap A_{u,v} \neq \emptyset \implies C_\mathcal{L}(u, U) \cap C_\mathcal{L}(v, U) = \emptyset$

2. $NM_\mathcal{L}(v, U) \cap A_{v,u} = \emptyset \land A_{v,u} \neq \emptyset \implies$

   $v \mid u, \quad C_\mathcal{L}(u, U) \subset C_\mathcal{L}(v, U) \land NM_\mathcal{L}(v, U) \cap E_{u,v} \subseteq NM_\mathcal{L}(u, U) \cap E_{u,v}$

   or

   $v \not\mid u, \quad C_\mathcal{L}(u, U) \cap C_\mathcal{L}(v, U) \land NM_\mathcal{L}(u, U) \cap A_{u,v} \neq \emptyset$
Example

Let $U = \{x_1 x_2^2, x_1, x_1^3 x_2\}$ (CF, D’Anna, Marotta’02). Then

$A_{1,2} = A_{3,2} = \emptyset$, $A_{1,3} = \{x_1\}$, $A_{2,1} = A_{3,1} = \{x_2\}$, $A_{2,3} = \{x_1, x_2\}$

$E_{1,2} = \{x_1\}$, $E_{1,3} = E_{3,2} = \emptyset$

and one of the possible involutive divisions restricted to $U$, in accordance to the theorem, is

$NM_1 = \{x_1\}$, $NM_2 = \{x_2\}$, $NM_3 = \emptyset$

whereas for Janet division

$NM_J(1) = \{x_1\}$, $NM_J(2) = \{x_1, x_2\}$, $NM_J(3) = \emptyset$

the involutive cone for the second element is narrower. Hence, the found restriction is better for $U$ than Janet division.

Open problem: extend this characterization to dynamical properties of involutive divisions determined by axiom (3).
Conclusion

- G.Carrà Ferro is one of founders of computer differential algebra. Her seminal paper on differential Gröbner bases is among most cited papers on differential algebra published over the last quarter of a century.
- Her research results on differential resultants are of considerable promise for the differential elimination with potentially many important applications.
- She formulated in 1994 a number of open problems in differential and related algebra most of which are still topical and reveal interesting research directions.
- Personally, I got a great pleasure from numerous discussions with Pina of the above presented and many other topics at our meetings and especially working together with her during my stay in Catania for a month in March 1999.
With Pina in Catania