Knaster-Tarski Theorem

Jayadev Misra

9/12/2014

This note presents a proof of the famous Knaster-Tarski theorem [1]. I have opted for clarity over brevity in the proof.

Complete Lattice A *complete lattice* is a partially ordered set $(L, \leq)$ in which every subset of L has a greatest lower bound (glb) and a least upper bound (lub) in L. The lub of the empty set is denoted by $\perp$ and glb of the empty set is $\top$. It follows that a complete lattice is never empty.

Monotonic Function over a Lattice Let f be a monotonic function over a lattice $(L, \leq)$, i,e, $x \leq y \;\Rightarrow\; f(x) \leq f(y)$, for any x and y in L. Point x is a (1) *prefixed* point if $f(x) \leq x$, (2) *postfixed* point if $x \leq f(x)$, and (3) *fixed* point if $f(x) = x$. Clearly, a fixed point is both a prefixed and a postfixed point.

Theorem [Knaster-Tarski]: For any complete lattice $(L, \leq)$,

1. The least fixed and the prefixed points of f exist, and they are identical.
2. The greatest fixed and the postfixed points of f exist, and they are identical.
3. The fixed points form a complete lattice.

Proof of (1) Let pre be the set of prefixed points, and p the glb of pre. Existence of p is guaranteed since L is a complete lattice. We show that p is both the least prefixed point and the least fixed point.

• p is the least prefixed point: For any prefixed point x,

$$
\begin{array}{ll}
 & p \leq x \\
\Rightarrow & \{f \text{ is monotonic}\} \\
 & f(p) \leq f(x) \\
\Rightarrow & \{x \text{ is a prefixed point; so, } f(x) \leq x\} \\
 & f(p) \leq x \\
\Rightarrow & \{\text{thus } f(p) \text{ is a lower bound of } pre. \text{ And, } p \text{ is the glb of } pre.\} \\
 & f(p) \leq p \\
\Rightarrow & \{\text{thus, } p \text{ is a prefixed point; and also a lower bound of } pre.\} \\
 & p \text{ is the least prefixed point}
\end{array}
$$

• p is the least fixed point: Since p is a prefixed point,

$$
\begin{array}{ll}
 & f(p) \leq p \\
\Rightarrow & \{f \text{ is monotonic}\} \\
 & f(f(p)) \leq f(p) \\
\Rightarrow & \{\text{from above, } f(p) \text{ is a prefixed point; and } p \text{ is a lower bound over } pre.\} \\
 & p \leq f(p) \\
\Rightarrow & \{p \text{ is a prefixed point; so } f(p) \leq p\} \\
 & p = f(p) \\
\Rightarrow & \{\ \ p \text{ is a fixed point. Also, every fixed point is a prefixed point, and} \\
 & \quad p \text{ is a lower bound over all prefixed points.} \\
 & \quad \text{So, } p \text{ is a lower bound over all fixed points}\} \\
 & p \text{ is the least fixed point}
\end{array}
$$

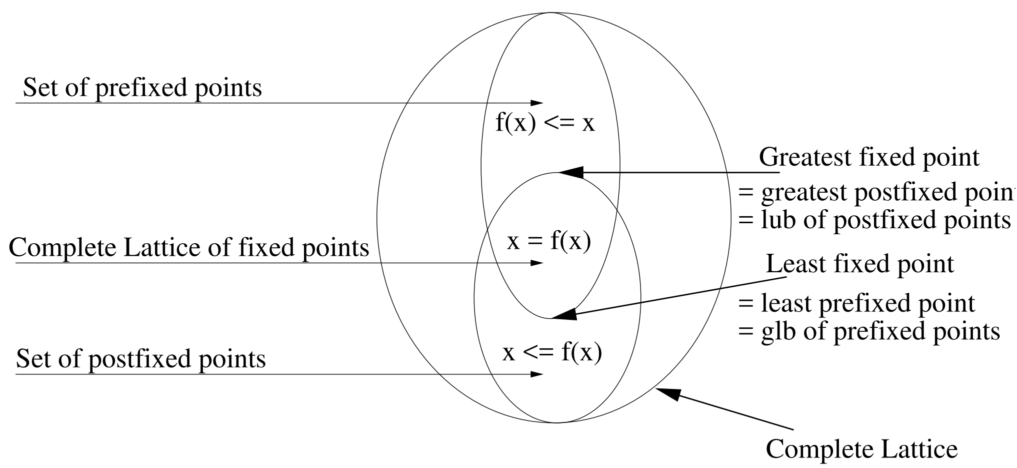

Figure 1: Pictorial Depiction of the Knaster-Tarski Theorem

Proof of (2) proof of (2) is dual of proof of (1), using lub for glb and postfixed points for prefixed points.

Proof of (3), The fixed points form a complete lattice: Let W be a subset of the fixed points. We show the existence of the supremum of W. Dually, W has an infimum, establishing that the fixed points form a complete lattice. Observe that the existence of least and greatest fixed points does not guarantee that W has a lub and glb.[1]

Let $q = lub(W)$ and $\widehat{W} = \{w \mid q \leq w\}$. Then $q \in \widehat{W}$ and $q = glb(\widehat{W})$.

3.1 $\widehat{W}$ is a complete lattice: $\widehat{W}$, being a subset of a complete lattice, has a lub and glb. We have $q = glb(\widehat{W})$ and $q \in \widehat{W}$; so, $glb(\widehat{W}) \in \widehat{W}$. Further, since $q \in \widehat{W}$, $q \leq lub(\widehat{W})$; therefore, $lub(\widehat{W}) \in \widehat{W}$ from the definition of $\widehat{W}$.

3.2 f maps $\widehat{W}$ to $\widehat{W}$: For any element w of W and x of $\widehat{W}$

$$
\begin{array}{ll}
 & w \leq q \text{ and } q \leq x \\
\Rightarrow & \{f \text{ monotonic: so } f(w) \leq f(q) \leq f(x)\} \\
 & f(w) \leq f(x) \\
\Rightarrow & \{W \text{ is a set of fixed points; so, } f(w) = w\} \\
 & w \leq f(x) \\
\Rightarrow & \{w \text{ is an arbitrary element of } W\} \\
 & lub(W) \leq f(x) \\
\Rightarrow & \{q = lub(W)\} \\
 & q \leq f(x) \\
\Rightarrow & \{\text{definition of } \widehat{W}\} \\
 & f(x) \in \widehat{W}
\end{array}
$$

3.3 From (3.1 and 3.2), f is a mapping over the complete lattice $\widehat{W}$. So, it has a least fixed point $\widehat{q}$ in $\widehat{W}$. Since q is the least element of $\widehat{W}$, $q \leq \widehat{q}$. Thus, $\widehat{q}$ is the supremum of W.

References

[1] A. Tarski. A lattice-theoretical fixpoint theorem and its application. *Pacific Journal of Mathematics*, 5:285–309, 1955.

[1] A lattice with a top and a bottom element is not necessarily a complete lattice. Here is a counterexample due to Vladimir Lifschitz. Let R be the set of rational numbers in the closed interval [0,1]; clearly R is a lattice under the standard order with a top and a bottom. For any irrational number x between 0 and 1 let I_x be the set of numbers in R that are less than x. Then I_x does not have a lub in R.