Consorzio COMETA - Progetto PI2S2

FESR

# Looking for Class Records in the 3x+1 problem by means of the Cometa grid

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# the 3x+1 Problem

misty origins:

- Collatz (1932) proposed another problem (of similar nature, still open), yet ...
- proposed by Thwaites (1952), with a 1000-pound prize
- "rediscovered" several times, whereby it is known under several names: Collatz, Syracuse, Kakutani, Hasse, Ulam, 3x+1, ...

"official" statement of the Problem: let $f : N_+ \to N_+$ be defined by the following rules:

$fx = 3x + 1$ if $x$ is odd

$fx = x/2$ if $x$ is even

is it true that repeated iteration of $f$ always converges to 1 ?

or, in other words, that $f^*$ has the $1 \to 4 \to 2 \to 1$ cycle as (unique) attractor?

"Mathematics is not yet ready for such problems" (Erdös)



# 3x+1 trajectories and records

this work does **not** aim at solving the 3x+1 Problem

a positive answer is conjectured

subject of interest here is the **dynamical behaviour** of $f^*$

basic concepts relating to the $f^*$ **dynamics:**

- **trajectory** at (origin) $x$ : the infinite sequence $x, fx, f^2x, \ldots$
- **delay** of (trajectory at) $x$ : the smallest $n$ such that $f^n x = 1$
- **delay class** $d$ : the set of those $x$ which have delay $d$

  remark : every delay class is populated ($2^d$ has delay $d$ )
- **class record (CR):** the smallest member of a delay class
- **delay record (DR):** an $x$ such that every $y < x$ has a lower delay

  fact : every DR is a CR (but the converse does not hold)

## distributed search of class records

the assessment that an x is a CR requires certainty that no lower y has the same delay

→ computation of **delay**(y) for all $y < x$ ?

not really, since several laws relate delays of joining trajectories

- **e.g.** for all x : **delay**(2x) = **delay**(x) + 1,
- and: **delay**(2x + 1) = **delay**(6x + 4) + 1 = **delay**(3x + 2) + 2

such laws may **speed-up** delay computation, yet CR finding does need **exhaustive search**

this is easily parallelized by partitioning the search space into disjoint intervals, explored by **independent jobs**, each producing its own set of **CR candidates**

smallests members, in the given interval, of each delay class

the final merge of distributed search results thus amounts to select the “best”, i.e. smallest, candidate for each delay class



## motivation for the search on the Cometa grid

a distributed search effort, to find 3x+1 class records, is active since quite a few years

it has found 2014 CRs so far, exploring the search space up to $57780 \cdot 10^{12}$

so, why taking the search to the COMETA grid?

4 good reasons:

- (proprietary) OS dependence
- machine architecture dependence (32-bit, vendor-specific)
- design of **novel optimization mechanisms**, afforded by 64-bit architecture
- simple parallelization structure (DAG partitioning), easy space-time tradeoff

and, last but not least:

**educational use:** parallel programming methodology, grid job control & monitoring

# a basic parallel algorithm

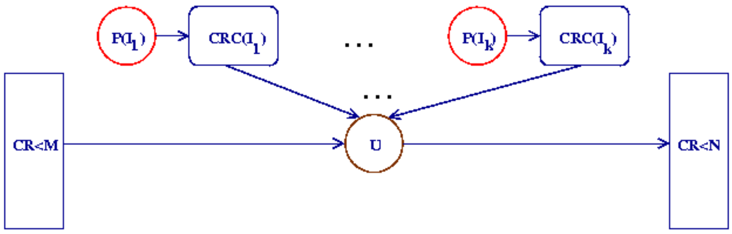
assume all CRs below M are known

**goal:** find all new CRs below N > M

- partition the search space [M, N [ into k pairwise disjoint intervals
- for each interval, find the best **CR candidates**, relatively to that interval

  that is, regardless of known CRs and of candidates from other intervals
- after all intervals are searched out, **merge** the parallel search outcomes

  also taking known CRs into account






# optimization (1): Sieving

the heart of CR finding is the **delay computation** function

its execution speed is **performance critical**

the fastest execution is that which... need not start :)

**coalescence** of trajectories entails that
**CRs fall outside certain congruence classes**

for example, 5 is the only CR in the congruence class 5 (mod 8)... why?

for n>0, the trajectories of 8n+5 and 8n+4 coalesce at 6n+4 after the same number of steps (3), so delay(8n+5) = delay(8n+4) → 8n+5 cannot be a CR

this generalizes to congruence classes $(2^{k-2} + (k \bmod 2)2^{k-1} - 1) \pmod{2^k}$, for k > 2

checking the general case? hardly efficient...

better off with a finite approximation (e.g. k=18), efficiently implemented by a **sieve**

## optimization (2): Tail cut-off

all trajectories leading to 1 (veritably all, thus) sooner or later fall below $2^t$, for any given, fixed **t**

delay computation may thus get quicker by storing all delay(n) for $n < 2^t$, and then adding delay(n) to the partially computed delay of x as soon as the trajectory starting at x reaches such an n

how to choose the tail cut-off threshold parameter **t** ?

by similar, architecture-dependent data as for the sieve size:

available RAM size ?

a surely lower threshold proves optimal (**t** between 12 and 16, usually), that depends on

**cache size !**



## optimization (3): Head cut-off

two DRs are **consecutive** if there is no DR in between them

if $x_1 < x_2$ is a pair of consecutive DRs, then **x** < $x_2$ → **delay(x) ≤ delay($x_1$)**

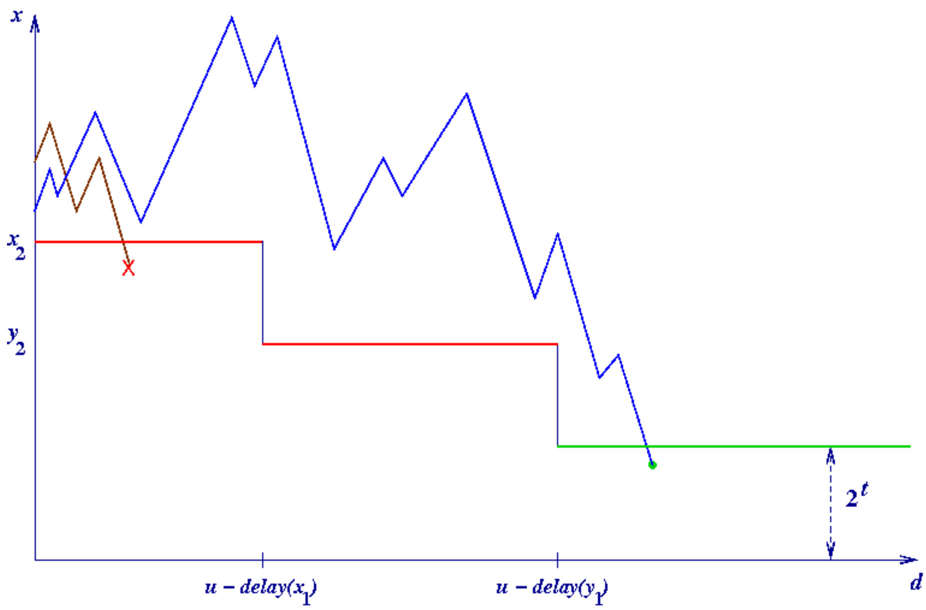
one can use this to stop the delay computation of "hopeless" trajectories ahead of time:

- assume all CRs below M are known
- let **u** be the smallest delay class whose CR is not below M
- **x** ≥ M may only be a CR if its delay is at least **u**
- if the trajectory of **x** falls below $x_2$ after **d** steps, then **delay(x) ≤ d + delay($x_1$)**
- hence, if **d + delay($x_1$) < u**, then **x** cannot be a CR

**virtue** of this technique: **"self-optimizing"**! CR search progress eventually leads to

- raise **u** → higher cut-off rate
- get higher pairs of consecutive DRs → earlier head cut-off

# Head and Tail cut-off in a picture






# optimization (4): Acceleration

a small acceleration of delay computation comes from replacing the function f with T, defined as f on the even numbers, whereas for **odd** x one has

$$Tx = (3x + 1)/2 = x + \lceil x/2 \rceil$$

clearly, if the T-trajectory from **x** to 1 has **D** applications of this rule and **E** applications of the halving rule, then **delay(x) = 2D + E**

the interest in T comes from the existence of a permutation of the residues (mod $2^k$), defined by the k-prefix of the so-called **parity vector** of T-trajectories, viz. the binary sequence, dependent on x, defined by $v_i(x) = (T^i x) \bmod 2$

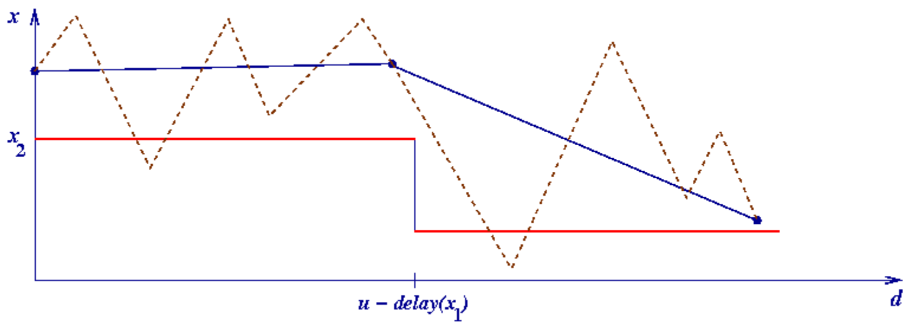
the k-prefix of $v_i(x)$ only depends on x (mod $2^k$)

thus one may define $2^k$ distinct k-step composites of T, and decide which applies to x by only looking at x (mod $2^k$) : not so small acceleration, by clever programming techniques

## interference and smoothening of Acceleration

optimal k for Acceleration depends on cache size, **but:**

unconstrained Acceleration reduces the effectiveness of Head cut-off!




solution: **smoothening** of Acceleration, that is, shorten Acceleration extent

- for those composites which may cause interference
- below appropriate, Head cut-off dependent thresholds



## performance figures

approximate average CPU time **h** (in hours) to search an interval of size $2^{44}$, in the neighbourhood of $2^{56}$:

| **u** | **h** |
|---|---|
| 1925 | 160 |
| 1951 | 150 |
| 1964 | 148 |
| 1985 | 124 |

**remark:** the latest performance gain is also due to the introduction of a new pair of consecutive DRs, which has nearly doubled the highest Head cut-off threshold

## state of the search and results

search started on 25/09/2007, from $2^{55} + 72 * 2^{48}$

- as of today, all CRs below $2^{56} + 14 * 2^{48}$ have been found, which means:
- 18 new CRs, including $R_{2000}$: challenge candidate confirmed
- one of them "truly new", that is, lower than the best known candidate until its discovery:

**Subject:** Re: R_1995 (real surprise! ;) [Re: last 4 CRs below 2^56]
**From:** Eric Roosendaal <eric@ericr.nl>
**Reply-To:** Eric Roosendaal <eric@ericr.nl>
**Date:** 30/11/2007 01:26
**To:** scollo@dmi.unict.it

>but R_1995 is better, really:
> 72230,523319,648959
Hey, that's great. I honestly didn't think that we would find any better candidates below 2000 anymore, and suddenly a new one still comes up. It's only a small improvement, but it's still an improvement.

So the search really yields new results. Excellent!



## questions?

Thank you very much for your kind attention!