Recurrence and attractors in state transition dynamics
A relational view

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relation algebra: standard concepts and notation

- **relation algebra:** a *complete, atomic* boolean algebra enriched with

  unary **converse**: $r^{\smile}$

  binary associative **composition**: $p\,;(q\,;r) = (p\,;q)\,;r$

  an **identity** constant: $1'\,; r = r\,; 1' = r$

  that satisfy some further **laws:**

  Schröder equivalences: $p\,;q \leq r \;\Leftrightarrow\; p^{\smile}\,;r^{-1} \leq q^{-1} \;\Leftrightarrow\; r^{-1}\,;q^{\smile} \leq p^{-1}$

  Tarski rule: $r \neq 0 \Rightarrow 1; r\,; 1 = 1$

- **provable laws** (among many others):

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- a relation algebra is *representable* if
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concepts and notation for state transition dynamics

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- **Kleene closure** and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

note: $q^+ = q^{\geq 1}$ and $q^* = q^{\geq 0}$.

- **monotypes:** subrelations of the identity, viz. $x \leq 1'$, such as
  domain of q: $\mathrm{dom}\, q = 1' \cdot (q \,;\, 1)$, **image** of q: $\mathrm{img}\, q = 1' \cdot (1 \,;\, q)$
- **atomic monotypes:** characterized by the quasiequations
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- **notation:** $x \lessdot y$ means that x is an atomic monotype and $x \leq y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define:**
  - $x \leq_q^{(n)} y \quad \text{if} \quad \mathrm{img}(x \,;\, q^{\geq n}) \leq y$
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  domain of q: $\mathrm{dom}\ q = 1' \cdot (q \,;\, 1)$, **image** of q: $\mathrm{img}\ q = 1' \cdot (1 \,;\, q)$
- **atomic monotypes:** characterized by the quasiequations
  $x \leq 1', \quad 1 \,;\, x \,;\, 1 = 1, \quad$ and $\quad y \leq x \wedge 1 \,;\, y \,;\, 1 = 1 \Rightarrow y = x$
- **notation:** $x \lessdot y$ means that x is an atomic monotype and $x \leq y$
- useful *higher-order binary relations on monotypes*: if q is a binary relation and x, y are monotypes in a relation algebra, **define:**
  - $x \leq_q^{(n)} y \quad$ if $\quad \mathrm{img}(x \,;\, q^{\geq n}) \leq y$
  - the **eventually below under q-iteration** relation on monotypes: $\precsim_q = \sum_{n \in \mathbb{N}} \leq_q^{(n)}$

concepts and notation for state transition dynamics

- **iteration:** $q^0 = 1', q^{i+1} = q \,;\, q^i$
- **Kleene closure** and the like:

$$q^* = \sum_{i \in \mathbb{N}} q^i, \quad q^+ = \sum_{i > 0} q^i, \quad q^{\geq n} = \sum_{i \geq n} q^i$$

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- **state transition (ST) dynamics**: (S, q)

  S : (discrete) **state space**

  $q \subseteq S \times S$: [total] transition **dynamics** [dom $q = 1'_S$]

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a simple model of epidemic propagation

as well as of unstable catalythic reaction:

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$C \longrightarrow C$

$G \longrightarrow \lambda$

$G \longrightarrow K$

$G \longrightarrow G$

- **instability** of agent G: it either dies or recovers (becoming immune: K)
- **states** : $(|C|+|K|, |G|)$
- **transitions** can be determined by equipping rules with a measure of **relative strength**: see [(Bianco *et al.*, 2006a), (Bianco *et al.*, 2006b), (Manca, 2006)]

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recurrence and attractors: basic definitions

Because of nondeterminism of the transition relation, concepts of **attracting set, recurrence, attractor** take *two distinct modal flavours*. Let b be a basin and $0 \neq a \leq b$:

- a is an *unavoidable attracting set* of b, a **□-attracts** b, if

$$b \leq \sum_{r \preceq_q a} r$$

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may hamper existence of the unavoidable attractor (see later)

Definition The q-dynamics is **finitary** if the q relation is image-finite on individual states, *i.e.* img $(x\,;q)$ represents a finite state whenever x represents an individual state

Lemma 1 Let b be a basin in a finitary q-dynamics. If there exists $x \leq b$ such that for no $n \in \mathbb{N}$ $\mathrm{img}(x\,;q^{\geq n}) \leq \mathrm{img}(r_\diamond\,;q^*)$, then there is a nonrecurrent x-flight in b

Proof idea:

- arrange the x-orbit in a x-rooted tree:
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Remark the q-finitarity hypothesis is fairly essential: consider an antiflight ξ, with ξ_0 the only fixed point in basin b, and an additional individual state $x \leq b$ with $\mathrm{img}(x\,;q) = \xi_{\mathbb{N}}$

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Lemma 1: existence of nonrecurrent flights

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Lemma 3: flights in absence of eternal recurrence

start everywhere

Lemma 3 If b is a basin in the q-dynamics with no eternally recurrent states, then every $x \leq b$ is the origin of a flight

Proof idea:

- for each x, find $x' \leq \mathrm{img}(x\,;q^{+}) \setminus \mathrm{img}(x\,;q^{\smile *})$ and a finite sequence of $n{+}2$ individual states $(\xi_i \mid 0 \leq i \leq n+1)$, for some $n \geq 0$, that satisfies the following requirements:
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- for each x, find $x' \leq \mathrm{img}(x\,;q^{+}) \setminus \mathrm{img}(x\,;q^{\smile *})$ and a finite sequence of $n+2$ individual states $(\xi_i \mid 0 \leq i \leq n+1)$, for some $n \geq 0$, that satisfies the following requirements:
  - (i) $\xi_0 = x$, $\xi_{n+1} = x'$, $\xi_{i+1} \leq \mathrm{img}(\xi_i\,;q)$, for $0 \leq i \leq n$;
  - (ii) $\xi_i \leq \mathrm{img}(x\,;q^{\smile *})$, for $0 < i \leq n$;
  - (iii) $\xi_i = \xi_j \Leftrightarrow i = j$, for $0 \leq i, j \leq n+1$.
- iterate the previous procedure and show injectivity of the resulting flight map

Lemma 3: flights in absence of eternal recurrence

start everywhere

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Theorem: recurrence and attractors

characterizes both *existence* and *extent* of attractors

Theorem In any basin b with the q-dynamics:

(i) $a_{\Box} = \mathrm{img}(r_{\Diamond}\, ; q^{*})$
if the q-dynamics is finitary and every flight is recurrent,
otherwise $a_{\Box} = 0$ if the converse $q^{\smile}$-dynamics is finitary
and if there is a nonrecurrent antiflight-free flight, under
the q-dynamics

(ii) $a_{\Diamond} = r_{\Box}$ if every flight is eternally recurrent,
otherwise $a_{\Diamond} = 0$

Remarks:

- finitarity assumptions are only needed for the characterization of the unavoidable attractor
- despite the structural difference, a certain analogy with Poincaré Recurrence Theorem surfaces, with boundedness and invariance replaced by finitarity and flight recurrence hypotheses

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research questions and perspectives

Moving from structureless to topological state spaces:

- definability of **weaker** notions of **recurrence**

  that is: replace **exact** occurrence of a state in its own trajectory with **approximate** occurrence

- definability of **weaker** notions of **attraction**

  that is: replace **exact** inclusion of orbits in the attracting set with **approximate** inclusion

- **generalization** of the characterization results presented here, linking recurrence and attractors, under the aforementioned weakenings

  at least for the **deterministic case**

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References: state transition dynamics

- V. Manca, G. Franco, G. Scollo
  State transition dynamics: basic concepts and molecular computing perspectives.
  Chapter 2 in: M. Gheorghe (Ed.)
  Molecular Computational Models: Unconventional Approaches
  Idea Group, Hershey, PA, USA (2005) 32–55.

- G. Scollo, G. Franco, V. Manca
  A relational view of recurrence and attractors in state transition dynamics.
  Proc. RelMiCS/AKA 2006
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  Springer, LNCS (2006) to appear.

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References: background for further study

P. Kůrka
Topological and symbolic dynamics.
Cours Spécialisés **11**,
Société Mathématique de France (2003).

G. Schmidt & Th. Ströhlein
Relations and Graphs
Discrete Mathematics for Computer Scientists,
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References: metabolic P systems

L. Bianco, F. Fontana, G. Franco, V. Manca
P Systems for Biological Dynamics.
in: G. Ciobanu, M.J. Pérez-Jiménez, G. Paun (Eds.)
Applications of Membrane Computing
Springer, Natural Computing Series (2006) 81–126.

L. Bianco, F. Fontana, V. Manca
P Systems with reaction maps.
Int'l J. of Foundations of Computer Science,
vol. 17:1, 2006, 27–48.

V. Manca
Topics and Problems in Metabolic P Systems.
Fourth Brainstorming on Membrane Computing
Sevilla, 2006.

References: metabolic P systems

- L. Bianco, F. Fontana, G. Franco, V. Manca
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