Brill-Noether theory for moduli spaces of sheaves on algebraic varieties

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Pragmatic 09
Let $X$ be an $n$-dimensional smooth projective variety over $K = \overline{K}$, $\text{char}(K) = 0$.

Let

$$M_{X,H}(r; c_1, \cdots, c_s)$$

be the moduli space of rank $r$, $H$-stable vector bundles $E$ on $X$ with fixed Chern classes

$$c_i(E) = c_i \quad \text{for} \quad i = 1, \cdots, s := \text{min}\{r, n\}.$$
Moduli spaces of stable vector bundles were constructed in the 1970’s by Maruyama.

Since then they have been extensively studied from different points of view.

Unfortunately, except in the classical case of vector bundles on curves, relatively little is known about their geometry in terms of the existence and structure of their subvarieties.
Deals with the case of line bundles on smooth projective curves $C$ of genus $g$.

It is concerned with the subvarieties $W^k$ of $Pic^d(C)$ whose line bundles have at least $k + 1$ independent global sections.

Basic questions concerning non-emptiness, connectedness, irreducibility, dimension, singularities....have been answered when the curve $C$ is generic on the moduli space of curves of genus $g$. 
Ways to generalize the classical Brill-Noether theory

Consider vector bundles of any rank on curves

- Giving rise to the Brill-Noether loci

\[ W^k(r, d) = \{ E \in M(r, d) | h^0(E) \geq k \} \]

in the moduli space of stable rank \( r \) and degree \( d \) vector bundles on curves.

- During the last two decades, a great amount of job has been made around this Brill-Noether stratification giving rise to nice and interesting descriptions of these subvarieties.

- Many questions concerning their geometry still remain open.

Consider line bundles on varieties of arbitrary dimension.
Ways to generalize the classical theory of Brill-Noether

Go in both directions simultaneously

- Consider $X$ a smooth projective variety of arbitrary dimension.
- Consider the moduli space $M_{X,H} (r; c_1, \cdots, c_s)$

  of rank $r$, $H$-stable vector bundles $E$ on $X$ with fixed Chern classes $c_i$.

- Try to study the subschemes in $M_{X,H} (r; c_1, \cdots, c_s)$ defined by conditions $\{ \dim H^j (X, E) \geq n_j \}$.
Remark:
If we are dealing with vector bundles $E$ on curves, we only have at most two non-vanishing cohomology groups $H^0(E)$ and $H^1(E)$ which are related by Riemann-Roch Theorem. Thus, the condition $h^0(E) \geq k$ defines a filtration of the moduli space $M(r, d)$.

If we are dealing with vector bundles $E$ on $n$-dimensional projective varieties, a priory we have $n + 1$ non-vanishing cohomology groups $H^i(E)$ and with the conditions $\{\dim H^i(X, E) \geq n_j\}$ we get a multigraded filtration.
The main GOAL is to introduce a Brill-Noether theory for moduli spaces of rank \( r \), \( H \)-stable vector bundles on algebraic varieties of arbitrary dimension, extending, in particular, the Brill-Noether theory on curves to higher dimensional varieties.
WHAT WE WILL DO:

- We will define the Brill-Noether locus

\[ W^k_H(r; c_1, \cdots, c_s) \]

in \( M_{X,H}(r; c_1, \cdots, c_s) \) as the set of vector bundles in \( M_{X,H}(r; c_1, \cdots, c_s) \) having at least \( k \) independent sections.

- Associated to this locus we will consider the generalized Brill-Noether number \( \rho_H^k(r; c_1, \cdots, c_s) \).

- We will prove that \( W^k_H(r; c_1, \cdots, c_s) \) has a natural structure of algebraic variety and that any of its non-empty components has dimension \( \geq \rho_H^k(r; c_1, \cdots, c_s) \).

- We will address the main problems and we will analyze them for several concrete moduli problems.
DEFINITION:
Let $H$ be an ample line bundle on $X$. For a torsion free sheaf $F$ on $X$ we set

$$
\mu(F) = \mu_H(F) := \frac{c_1(F)H^{n-1}}{rk(F)}.
$$

The sheaf $F$ is said to be $H$-semistable if

$$
\mu_H(E) \leq \mu_H(F)
$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then $F$ is $H$-stable.

Remark:
The definition of stability depends on the choice of the ample line bundle $H$. 
THEOREM:
Let $X$ be an $n$-dimensional smooth projective variety and $M_H = M_{X,H}(r; c_1, \cdots, c_s)$. Assume that for any $E \in M_H$, $H^i(E) = 0$ for $i \geq 2$. Then, for any $k \geq 0$, there exists a determinantal variety $W^k_H(r; c_1, \cdots, c_s)$ such that

$$\text{Supp}(W^k_H(r; c_1, \cdots, c_s)) = \{ E \in M_H | h^0(E) \geq k \}.$$

Moreover, each non-empty irreducible component of $W^k_H(r; c_1, \cdots, c_s)$ has dimension at least $\dim(M_H) - k(k - \chi(r; c_1, \cdots, c_s))$, and $W^{k+1}_H(r; c_1, \cdots, c_s) \subset \text{Sing}(W^k_H(r; c_1, \cdots, c_s))$ whenever $W^k_H(r; c_1, \cdots, c_s) \neq M_{X,H}(r; c_1, \cdots, c_s)$. 
Sketch of the Proof.

Assume that $M_H$ has a universal family $\mathcal{U} \to X \times M_H$ such that for any $t \in M_H$,

$$\mathcal{U}|_{X \times \{t\}} = E_t$$

is an $H$-stable rank $r$ vector bundle on $X$ with Chern classes $c_i$. Let $D$ be an effective divisor on $X$ such that for any $t \in M_H$,

$$h^0(E_t(D)) = \chi(E_t(D)), \quad H^i(E_t(D)) = 0, \quad i \geq 1.$$

Consider $\mathcal{D} = D \times M_H$ the corresponding product divisor on $X \times M_H$ and denote by

$$\nu : X \times M_H \to M_H$$

the natural projection.
We get the exact sequence
\[
0 \to \nu_* \mathcal{U} \to \nu_* \mathcal{U}(\mathcal{D}) \xrightarrow{\gamma} \nu_* (\mathcal{U}(\mathcal{D})/\mathcal{U}) \to R^1 \nu_* \mathcal{U} \to 0.
\]

The map \(\gamma\) is a morphism between locally free sheaves on \(M_H\) of rank \(\chi(E_t(D))\) and \(\chi(E_t(D)) - \chi(E)\) respectively.

The \((\chi(E_t(D)) - k)\)-th determinantal variety

\[
W^k_H(r; c_1, \cdots, c_s) \subset M_H
\]

associated to it has support

\[
\{ E_t \in M_H | \text{rank} \gamma_{E_t} \leq \chi(E_t(D)) - k \}
\]

i.e. \(W^k_H(r; c_1, \cdots, c_s)\) is the locus where the fiber of \(R^1 \nu_* \mathcal{U}\) has dimension at least

\[
(\chi(E_t(D)) - \chi(E_t)) - (\chi(E_t(D)) - k) = k - \chi(E_t).
\]

For any \(E_t \in M_H\) the assumption \(h^i(E_t) = 0, i \geq 2,\) implies

\[
h^1(E_t) = h^0(E_t) - \chi(E_t).
\]
Thus,

\[
\text{Supp}(W^k_H(r; c_1, \cdots, c_s)) = \{ E \in M_H \mid h^1(E) \geq k - \chi(E) \} = \{ E \in M_H \mid h^0(E) \geq k \}.
\]

Finally, since \(W^k_H(r; c_1, \cdots, c_s)\) is a \((\chi(E_t(D)) - k)\)-determinantal variety associated to a morphism between locally free sheaves of rank \(\chi(E_t(D))\) and \(\chi(E_t(D)) - \chi(E)\) respectively, any of its non-empty irreducible components has dimension greater or equal to \(\dim(M_H) - k(k - \chi(E))\) and

\[
W^{k+1}_H(r; c_1, \cdots, c_s) \subset \text{Sing}(W^k_H(r; c_1, \cdots, c_s))
\]

whenever \(W^k_H(r; c_1, \cdots, c_s) \neq M_{X,H}(r; c_1, \cdots, c_s)\).
Remark: The cohomological assumptions are natural if we want a filtration of $M_H$ by the subvarieties $W^k_H(r; c_1, \cdots, c_s)$.

Any vector bundle $E$ on $X$ has $n + 1$ cohomological groups and one is forced to look for a multigraded filtration of $M_H$ by the sets $\{ E \in M_H | h^i(E) \geq k_i \}$.

Under our cohomological assumptions,

$$\dim H^0(E) - \dim H^1(E) = \chi(E) = \chi(r; c_1, \cdots, c_s).$$

Hence, it makes sense to consider only the filtration of the moduli space $M_H$ by the dimension of the space of global sections.

Instanton bundles on $\mathbb{P}^{2n+1}$, Schwarzenberger and Steiner bundles on $\mathbb{P}^n$, Steiner and Spinor bundles on $Q_n \subset \mathbb{P}^{n+1}$ and many others satisfy these cohomological conditions.
COROLLARY:

Let $X$ be a smooth projective surface and assume that

$$c_1 H \geq r K_X H.$$ 

Then, for any $k \geq 0$, there exists a determinantal variety $W^k_H(r; c_1, c_2)$ such that

$$\text{Supp}(W^k_H(r; c_1, c_2)) = \{ E \in M_H | h^0(E) \geq k \}.$$ 

Moreover, each non-empty irreducible component of $W^k_H(r; c_1, c_2)$ has dimension greater or equal to

$$\rho^k_H(r; c_1, c_2) = \dim(M_H) - k(k - r(1 + P_a(X)) + \frac{c_1 K_X}{2} - \frac{c_1^2}{2} + c_2).$$
**DEFINITION:**

The variety $W^k = W^k_H(r; c_1, \cdots, c_s)$ is called the \textit{k-Brill-Noether locus} of $M_H$ and

$$\rho^k = \rho^k_H(r; c_1, \cdots, c_s) := \dim M_H - k(k - \chi(r; c_1, \cdots, c_s))$$

is called the \textit{generalized Brill-Noether number}.

\textbf{Remark:} When $X$ is a smooth projective curve and we consider the moduli space $Pic^d(X)$ of degree $d$ line bundles on $X$, then we recover the classical Brill-Noether loci and the generalized Brill-Noether number is the classical Brill-Noether number

$$\rho = \rho(g, r, d) = g - (r + 1)(g - d + r)$$
Remark:

The Brill-Noether locus $W^k$ has dimension greater or equal to $\rho^k$ and the number $\rho^k$ is also called the expected dimension of the corresponding Brill-Noether locus.

QUESTION:

Whether the dimension of $W^k_{H}(r; c_1, \cdots, c_s)$ and its expected dimension coincide provided $W^k_{H}(r; c_1, \cdots, c_s) \neq \emptyset$
QUESTION:

- Whether $\rho^k < 0 \Rightarrow W^k = \emptyset$ ?
- Whether $\rho^k \geq 0 \Rightarrow W^k \neq \emptyset$ ?
- Whether $\rho^k \geq 0$ and $W^k \neq \emptyset$ implies

\[ \rho^k = \rho^k_H(r; c_1, \cdots, c_s) = \dim W^k_H(r; c_1, \cdots, c_s) \]

If we deal with varieties of higher dimension, a great number of different situations and pathologies appear and this makes this new theory and emerging field of interest.
EXAMPLE:

Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and denote by $l_1, l_2$ the generators of $Pic(X)$. For any $n \geq 2$ we fix the ample line bundle

$$L = l_1 + nl_2.$$ 

For any $k$, $1 \leq k \leq n$, such that $8n - 3 < k(k - 1)$

$$W^k_L(2; (2n - 1)l_2, 2n) \subset M_{X,L}(2; (2n - 1)l_2, 2n)$$

is non-empty and the generalized Brill-Noether number,

$$\rho^k_L(2; (2n - 1)l_2, 2n) = 8n - 3 - k(k - 1),$$

is negative.
Brill-Noether theory on Hirzebruch surfaces

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Brill-Noether theory for moduli spaces of sheaves
Brill-Noether theory on Hirzebruch surfaces

For any integer $e \geq 0$, let $X_e \cong \mathbb{P}(E) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a non singular, Hirzebruch surface.

We denote by $C_0$ and $F$ the standard basis of $\text{Pic}(X_e) \cong \mathbb{Z}^2$ such that $C_0^2 = -e$, $F^2 = 0$ and $C_0F = 1$.

In this basis $-K_{X_e} = 2C_0 + (e + 2)F$ and a divisor $L = aC_0 + bF$ on $X_e$ is ample if, and only if, it is very ample, if and only if, $a > 0$ and $b > ae$, and that $D = a'C_0 + b'F$ is effective if and only if $a' \geq 0$ and $b' \geq 0$. 

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Brill-Noether theory for moduli spaces of sheaves
Our plan is to explain you results concerning:

- Non-emptiness of Brill-Noether loci of stable vector bundles on \( X_e \).
- Emptiness of Brill-Noether loci of stable vector bundles on \( X_e \).
Non-emptiness of some Brill-Noether loci of stable vector bundles on $X_e$

We fix our attention into the following moduli space

$$M_H(r; C_0 - xF, c_2)$$

where $x$, $r$, $c_2$ are integers with $x > 0$, $r \geq 2$ and $c_2 \gg 0$ and we have fixed the ample divisor

$$H := C_0 + (x + e + 1)F.$$

Remark:

Under the assumptions $x > 0$ and $c_2 \gg 0$, we have

$$\rho^k_H(r; C_0 - xF, c_2) < 0 \quad \text{for} \quad k \geq r - 1.$$ 

So, we will study the Brill-Noether loci $W^k_H(r; C_0 - xF, c_2)$ of the Brill-Noether stratification of $M_H(r; C_0 - xF, c_2)$ for $k$ in the range $1 \leq k \leq r - 2$. 

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THEOREM:

Let $X_e$ be a Hirzebruch surface. Then for any integer $k$, $1 \leq k \leq r - 2$, and $c_2$ such that $c_2 \equiv 0 \mod (r - k - 1)$ there exists an irreducible component of the Brill-Noether loci

$$W^k_H(r; C_0 - xF, c_2)$$

of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension $\rho^k_H(r; C_0 - xF, c_2)$.

Idea:

We will construct a family $\mathcal{F}$ of rank $r$ vector bundles $E$ given by a non-trivial extension

$$0 \rightarrow \mathcal{O}^k_{X_e} \rightarrow E \rightarrow G \rightarrow 0$$

where $G$ sits in an open subset

$\mathcal{H}^0(r - k; C_0 - xF, c_2) \subset M_H(r - k; C_0 - xF, c_2)$.

Then check that $E$ is stable and compute the dimension of $\mathcal{F}$. 

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Definition:
Let $X_e$ be a Hirzebruch surface. A coherent sheaf $E$ on $X_e$ is said to be prioritary if it is torsion free and if

$$\text{Ext}^2(E, E(-F)) = 0.$$ 

The following Lemma is the key point in order to be able to use prioritary sheaves to study moduli spaces of stable vector bundles.

Lemma:
Let $X_e$ be a Hirzebruch surface and let $H$ be an ample divisor on $X_e$. Then any $H$-semistable, torsion free sheaf $E$ on $X_e$ is prioritary.
Proposition:

Let $X_e$ be a Hirzebruch surface and let $H$ be an ample divisor on $X_e$. Fix integers $\alpha$, $x$ and $c$ with $\alpha, x > 0$ and $c < 0$. If $c \ll 0$, then a generic vector bundle $E \in M_H(\alpha + 1; C_0 - xF, -\alpha c)$ sits in an exact sequence of the following type

$$0 \to \mathcal{O}_{X_e}(C_0 - (x - \alpha c)F) \to E \to \mathcal{O}_{X_e}(-cF)^{\alpha} \to 0. \quad (1)$$
THEOREM:

Let $X_e$ be a Hirzebruch surface. Then for any integer $k$, $1 \leq k \leq r - 2$ such that $c_2 \equiv 0 \mod (r - k - 1)$ there exists an irreducible component of the Brill-Noether loci $W^k_H(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension $\rho^k_H(r; C_0 - xF, c_2)$.
Sketch of the Proof.

Fix $k$ an integer, $1 \leq k \leq r - 2$, such that $(r - k - 1)$ divides $c_2$ and denote by $f$ the negative integer such that $c_2 = -(r - k - 1)f$.

Let $\mathcal{F}$ be the irreducible family of rank $r$ vector bundles $E$ given by a non-trivial extension

$$0 \rightarrow \mathcal{O}_{X_e}^k \rightarrow E \rightarrow G \rightarrow 0 \quad (2)$$

where $G$ sits in the open subset

$\mathcal{H}^0(r - k; C_0 - xF, c_2) \subset M_{\mathcal{H}}(r - k; C_0 - xF, c_2)$ given above.

First of all notice that any $E \in \mathcal{F}$ is a rank $r$ vector bundle with Chern classes

$$(c_1(E), c_2(E)) = (C_0 - xF, -(r - k - 1)f) = (C_0 - xF, c_2)$$

and by construction $h^0(E) \geq k$. 
Claim: $E$ is $H$-stable.

Proof of the Claim: We proceed by induction on $k$.

In case $k = 1$, $E$ is given by a non-trivial extension

$$0 \rightarrow \mathcal{O}_{X_e} \rightarrow E \overset{\sigma}{\rightarrow} G \rightarrow 0$$

where $G$ is a rank $r - 1$, $H$-stable vector bundle. Checking the definition of stability we see that $E$ is $H$ stable.
Now, we fix $k > 1$ and we consider the following commutative diagram of vector bundles

$$
0 \longrightarrow \mathcal{O}_{X_e} \longrightarrow \mathcal{O}_{X_e}^k \longrightarrow \mathcal{O}_{X_e}^{k-1} \longrightarrow 0
$$

$$
0 \longrightarrow \mathcal{O}_{X_e} \longrightarrow E \longrightarrow \overline{E} \longrightarrow 0.
$$

By hypothesis of induction, $\overline{E}$ is $H$-stable and thus by the first case $k = 1$, $E$ is $H$-stable which concludes the proof of the Claim.
Hence, the family $\mathcal{F}$ defines a non-empty irreducible component of the Brill-Noether loci $W^k_H(r; C_0 - xF, c_2)$.

We compute its dimension and see that

$$\dim \mathcal{F} = \dim \mathcal{H}^0(r - k; C_0 - xF, c_2) + \dim \text{Grass}(k, h)$$

being $h = \text{ext}^1(G, \mathcal{O}_{\mathcal{X}_e})$ and $\text{Grass}(k, h)$ the Grassmann variety of $k$-dimensional linear subspaces of $\text{Ext}^1(G, \mathcal{O}_{\mathcal{X}_e})$ coincides with $\rho^k_H(r; C_0 - xF, c_2)$

Therefore, $\mathcal{F}$ is an irreducible component of the Brill-Noether loci $W^k_H(r; C_0 - xF, c_2)$ of the moduli space $M_H(r; C_0 - xF, c_2)$ which has the expected dimension, namely, $\rho^k_H(r; C_0 - xF, c_2)$. 

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THEOREM:
Let $X$ be a smooth algebraic surface, $H$ an ample divisor s.t. $K_X H \leq 0$ and $E$ a semistable rank $r \geq 2$ vector bundle. Set $a := \left\lceil \frac{(r^2 - 1)H^2}{2} \right\rceil$ or $a = 2H^2$ if $r = 2$. If $0 \leq c_1(E)H < arH^2 + rK_X H$, then

$$h^0(E) \leq r + \frac{ac_1(E)H}{2}.$$

COROLLARY:
Let $X$ be a smooth algebraic surface, $H$ an ample divisor on $X$ such that $K_X H \leq 0$. Let $r \geq 2$, $c_2 \gg 0$ and set $a := \left\lceil \frac{(r^2 - 1)H^2}{2} \right\rceil$ or $a = 2H^2$ if $r = 2$. Assume that $0 \leq c_1(E)H < arH^2 + rK_X H$, then

$$W^k_H(r; c_1, c_2) = \emptyset.$$
Let $C \in |aH|$ be a general smooth connected curve. Since

$$\left(\frac{a+2}{2}\right) - \frac{a-1}{a} > \deg(X) \cdot \max\left\{\frac{r^2 - 1}{4}, 1\right\},$$

by Flenners restriction theorem, $E|_C$ is a rank $r$ semistable vector bundle on $C$ of degree equal to $ac_1(E)H$.

On the other hand, by the adjunction formula

$$2g(C) - 2 = C(C + K_X) = aH(aH + K_X) = a^2H^2 + aHK_X.$$

Hence,

$$0 \leq \mu(E|_C) = \frac{ac_1(E)H}{r} \leq a^2H^2 + aHK_X = 2g(C) - 2$$

and therefore, applying Clifford’s Theorem for semistable vector bundles on curves, we have

$$h^0(E|_C) \leq r + \frac{ac_1(E)H}{2}.$$
To finish, we only need to check that

\[ h^0(E) \leq h^0(E|_C). \]

To this end, we tensor by \( E \) the short exact sequence

\[ 0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0 \]

and taking cohomology we get

\[ 0 \rightarrow H^0(E(-C)) \rightarrow H^0(E) \rightarrow H^0(E|_C) \rightarrow \cdots. \tag{3} \]

If \( H^0(E(-C)) \neq 0 \), then \( \mathcal{O}_X(C) \hookrightarrow E \) and since \( E \) is semistable with respect to \( H \) we have

\[ CH = aH^2 \leq \frac{c_1(E)H}{r} < aH^2 + K_XH \]

which contradict the fact that \( K_XH \leq 0 \). Therefore, \( H^0(E(-C)) = 0 \) and from the exact sequence (3) we deduce \( h^0(E) \leq h^0(E|_C) \) and we finish the proof.
COROLLARY:

Let $x, r, c_2$ with $x > 0$, $r \geq 2$ and $c_2 \gg 0$. Let $H := C_0 \left( x + e + 1 \right) F$ be an ample divisor on $X_e$ and set $a := \lceil \frac{(r^2 - 1)H^2}{2} \rceil$ or $a = 2H^2$ if $r = 2$. Then, for any $k > r + \frac{a}{2}$

$$W^k_H(r; C_0 - xF, c_2) = \emptyset.$$