THE SPACE OF COMPACT OPERATORS CONTAINS $c_0$ WHEN A NONCOMPACT OPERATOR IS SUITABLY FACTORIZED*

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In this note we generalize certain results on when the space $K(X, Y)$ of compact operators contains an isomorphic copy of the sequence space $c_0$, a fact strictly connected to the nonexistence of a projection from the space $L(X, Y)$ onto the subspace $K(X, Y)$ as showed in the papers [3], [6]. One of the first results in this direction was obtained by Kalton in [7] who proved that if there is a noncompact operator with a domain space $X$ possessing an unconditional finite dimensional expansion of the identity and taking values in an arbitrary Banach space $Y$ then $c_0$ embeds into $K(X, Y)$. Diestel and Morrison [1] have proved the same statement under the assumption that $Y$ has an unconditional basis. Other results of the same nature obtained by Feder in [5], have been generalized by the authors in the recent paper [4]; in particular, it was there shown that if $L_{w^*}(X^*, Y)$ contains a noncompact operator, if the space $Y$ has the compact approximation property and if $Y \subset Y_1$ where the space $Y_1$ has an unconditional expansion of the identity, then again $c_0 \subset K_{w^*}(X^*, Y)$ (here $L_{w^*}(X^*, Y)$ denotes the space of $w^*-w$ continuous operators). Another similar result is contained in [2] where the first author proved that if there is a non compact operator factorizing through a reflexive Banach space with an unconditional basis then again $c_0$ embeds into $K(X, Y)$.

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In this note we show that all these results, as well as other facts from [5], actually are consequence of our Proposition 1. It describes a quite general procedure useful to construct copies of $c_0$ inside $K(X,Y)$ when starting from the existence of non compact operators.

We observe that the proof of our Proposition 1 below actually is a refinement of the techniques used in the previous papers; but even if not original at all, it allows us to cover (in the separable case) the old quoted results and to furnish some new facts; among them Theorem 1 is, in our opinion, the main new application.

Before finishing this Introduction we remark that in [2] and [6] it was independently shown that if a noncompact operator $T \in L(X,Y)$ factorizes through a Banach space which has an unconditional basis then $c_0 \subset K(X,Y)$; this seems to be the only old result not covered by the present ones.

Before presenting the main result we need a definition.

**Definition.** We shall say that \( \{K_n\} \subset K(X) \) is an unconditional compact approximating sequence if \( \|A_n x - x\| \to 0 \) and the sum \( \sum (A_{n+1} - A_n)x \) is weakly unconditionally Cauchy for all \( x \in X \). Moreover, we shall say that such a sequence is shrinking if \( \{K_n^*\} \) is an (unconditional) compact approximating sequence for \( X^* \).

A Banach space \( X \) is said to have the (shrinking) unconditional compact approximation property if there is a (shrinking) unconditional compact approximating sequence for \( X \).

In fact the above definition is possible for nets also, but in this section sequences are substantial.

We shall use also the following refinement of a fact due to Kalton [7]:

**Fact (K).** Let \( \hat{X} \subset X^* \) be total on \( X \), let \( \hat{Y} \subset Y^* \) be a norming subspace and suppose that the sequence \( \{T_n\} \subset K(X,Y) \) has the property that \( \hat{y}(T_n x) \to 0 \) for all \( x \in X \) and all \( \hat{y} \in \hat{Y} \). Suppose further that the unit ball \( B_X \) of \( X \) is \( w(X, \hat{X}) \) compact, the unit ball \( B_{\hat{Y}} \) of \( \hat{Y} \) is \( w(\hat{Y}, Y) \) compact and that the \( T_n \)'s are \( w(X, \hat{X}) \)-\( w(Y, \hat{Y}) \) continuous. Then \( T_n \to 0 \) in the weak topology of the space \( L(X,Y) \).

**Proof.** Suppose \( T \in K(X,Y) \) is \( w(X, \hat{X}) \)-\( w(Y, \hat{Y}) \) continuous. Then consider the compact topological space \( K = B_X \times B_{\hat{Y}} \) where on \( B_X \) we consider the \( w(X, \hat{X}) \) topology and on \( B_{\hat{Y}} \) the \( w(\hat{Y}, Y) \)-topology. Let \( f(x, \hat{y}) = \hat{y} T(x) \) define a real function on \( K \). Using the fact that on \( \overline{T(B_X)} \) the norm topology and the Hausdorff topology \( w(Y, \hat{Y}) \) coincide, it is not difficult to prove that \( f \) is continuous. So we may consider \( T_n \) as continuous functions on the compact Hausdorff space \( K \) equipped with the described topology. Then our convergence assumption and the Lebesgue theorem imply that \( T_n \to 0 \) in the weak topology of the normed space \( C(K) \) and thus also in the weak topology of the Banach space \( K(X,Y) \).
We are now ready to give our

**Proposition 1.** Let \( T \in L(X,Y) \) be an operator and let \( T = BA \) be a factorization of \( T \) through a Banach space \( E \). Suppose that \( E \) is isomorphic to a subspace of a Banach space \( E_1 \) by an isomorphism \( J \). Suppose also that \( \hat{X} \) is a total subspace of \( X^* \). Finally, let the following conditions (i)--(vi) be satisfied:

(i) there are a Banach space \( Y_1 \) containing isomorphically \( Y \) by an isomorphism \( I \) and a bounded linear operator \( \tilde{B} : E_1 \to Y_1 \), that is an extension of the operator \( B : E \to Y \) in the sense that \( \tilde{B} J(e) = IB(e) \) for all \( e \in E \),

(ii) there are a norming subspace \( \hat{Y}_1 \) of \( Y_1^* \) and continuous operators \( B_n \in K(E_1) \) for all \( n \in \mathbb{N} \), such that

\[
\hat{y}_1 \left( \sum_{i=1}^{n} \tilde{B} B_i J A(x) \right) \overset{n}{\longrightarrow} \hat{y}_1 (\tilde{B} J A(x)) \text{ for all } x \in X \text{ and all } \hat{y}_1 \in \hat{Y}_1
\]

and such that \( \sum B_n \) is a weakly unconditionally Cauchy (WUC) series in the space \( L(E_1) \),

(iii) there is a sequence \( \{A_n\} \subset K(E) \) of continuous operators such that, for all \( x \in X \), \( IBA_n A(x) \to IBA(x) \) in the \( w(Y_1, \hat{Y}_1) \)-topology,

(iv) \( IBA_A \colon X \to Y_1 \) is \( w(X, \hat{X}) \)-\( w(Y_1, \hat{Y}_1) \) continuous,

(v) \( \tilde{B} B_i J A : X \to Y_1 \) is \( w(X, \hat{X}) \)-\( w(Y_1, \hat{Y}_1) \) continuous,

(vi) the unit balls of the spaces \( X \) and \( \hat{Y}_1 \) are compact in the \( w(X, \hat{X}) \) and \( w(\hat{Y}_1, Y_1) \) topologies, respectively.

Then certain convex blocks of \( \{IBA_A\} \) are (WUC) and, in each point \( x \in X \), they converge to \( IT(x) \) in the \( w(Y_1, \hat{Y}_1) \) topology.

Moreover, if the operator \( T \) is not compact then the sequence space \( c_0 \) is isomorphically contained in \( \text{span} \{BA_A\} \subset K(X,Y) \).

**Proof.** The conditions (ii) and (iii) give that for all \( x \in X \) and all \( \hat{y}_1 \in \hat{Y}_1 \) we have

\[
\hat{y}_1 (IBA_n A(x)) \overset{n}{\longrightarrow} \hat{y}_1 (IBA(x)) \tag{1}
\]

and

\[
\hat{y}_1 \left( \sum_{i=1}^{n} \tilde{B} B_i J A(x) \right) \overset{n}{\longrightarrow} \hat{y}_1 (\tilde{B} J A(x)). \tag{2}
\]

Thus, since \( \tilde{B} \) extends \( B \) in the sense quoted in the assumption (ii), we get easily

\[
\hat{y}_1 (IBA_n A(x)) - \hat{y}_1 \left( \sum_{i=1}^{n} \tilde{B} B_i J A(x) \right) \overset{n}{\longrightarrow} 0. \tag{2}
\]
Now from (iv)-(v) we see that the operators
\[ IBnA - \sum_{i=1}^{n} \tilde{B}B_iJA : X \to Y_1 \]
are \( w(X, \tilde{X})-w(Y_1, \tilde{Y}_1) \) continuous; so we may deduce from (vi), (2) and Fact (K) that
\[ U_n = IBnA - \sum_{i=1}^{n} \tilde{B}B_iJA \xrightarrow{n \to \infty} 0 \]
in the weak topology of the space \( K(X, Y_1) \).

Now we proceed as in [9, p. 32]. Since \( U_n \xrightarrow{w} 0 \), we can find disjoint convex combinations (blocks) \( U'_j \) of \( \{U_n\} \), such that \( \sum_{j=1}^{\infty} ||U'_j|| < \infty \). Let \( Y'_j \) be the blocks of \( \{Y_n\} = \{BA_nA\} \) built with the same coefficients and let us put \( Z_j = Y_{j+1}' - Y'_j \). Computing, we get that
\[ IZ_j = U_{j+1}' - U'_j + C'_j, \]
where \( C'_j \)'s are disjoint blocks of \( \{C_n\} = \{\tilde{B}B_nJA\} \) with coefficients between 0 and 1.

Now we claim that \( \sum_{j=1}^{\infty} IZ_j \) is a weakly unconditionally Cauchy (WUC) series. To see this let \( Z^* \in K(X, Y_1)^* \). Then we have
\[ \sum_{j=1}^{\infty} |Z^*(IZ_j)| \leq 2||Z^*|| \cdot \sum_{j=1}^{\infty} ||U'_j|| + \sum_{n=1}^{\infty} |Z^*(C_n)| < \infty \]
using the fact that \( \sum_{j=1}^{\infty} C_n \) is a WUC series thanks to (ii). Indeed, (ii) means that
\[ || \sum_{n=1}^{m} \pm B_n || \leq K \text{ for all } m \text{ and all } \pm \text{ and thus } \{ || \sum_{n=1}^{m} \pm C_n || ; m \in N \} \text{ is also bounded}, \]
meaning that \( \sum C_n \) is WUC. But \( I \) is an isomorphism; so we conclude that also \( \sum_{j=1}^{\infty} Z_j \) is a WUC series. Further we observe that \( \sum_{j=1}^{\infty} Z_j \) is not norm convergent. Indeed, (1) may be rewritten
\[ \tilde{g}_1(IY_n(x)) \xrightarrow{n \to \infty} \tilde{g}_1(IT(x)) \quad \text{for } \tilde{g}_1 \in \tilde{Y}_1, x \in X \]
which implies that also for convex blocks \( Y'_j \) we have
\[ \tilde{g}_1(IY'_n(x)) \xrightarrow{n \to \infty} \tilde{g}_1(IT(x)). \]

Now assume that \( T \) is not compact; it easily follows that the sequence \( \{Y'_n\} \subset K(X, Y) \) does not converge in the norm topology since otherwise, by (3), \( \{IY'_n\} \) would converge (in the norm) to the non compact operator \( IT \). The famous Bessaga-Pelczyński Theorem (see [8]) now ensures that a subsequence of \( \{Z_j\} \) is equivalent to the unit vector basis of \( c_0 \), which finishes the proof. \( \square \)
As a special case we might formulate

**Proposition 1a.** Let \( T \in L(X, Y) \) be an operator and let \( T = BA \) be a factorization of \( T \) through a Banach space \( E \). Suppose that \( E \) is isomorphic to a subspace of a Banach space \( E_1 \) by an isomorphism \( J \) and that, further, \( E \subseteq E^* \) and \( E_1 \subseteq E_1^* \) are subspaces such that \( J \) is \( w(E, E_1) \)-\( w(E_1, E_1) \) continuous. Suppose also that \( \hat{Y} \) is a subspace of \( Y^* \), \( \hat{X} \) a total subspace of \( X^* \). Finally, let the following conditions (i)–(vi) be satisfied:

(i) there are a Banach space \( Y_1 \) containing isomorphically \( Y \) by an isomorphism \( I \), a norming subspace \( \hat{Y}_1 \) of \( Y_1^* \) such that \( I \) is \( w(Y, \hat{Y}) \)-\( w(Y_1, \hat{Y}_1) \) continuous and a \( w(E_1, E_1) \)-\( w(Y_1, \hat{Y}_1) \) continuous bounded linear operator \( B : E_1 \to Y_1 \), that is an extension of the operator \( B : E \to Y \) in the sense that \( BJ(e) = IB(e) \) for all \( e \in E \),

(ii) there are \( w(E_1, E_1) \)-continuous operators \( B_n \in K(E_1) \) for all \( n \in N \), such that \( \sum_{n=1}^{\infty} \hat{z}_1(B_n(z_1)) = \hat{z}_1(z_1) \) for all \( z_1 \in E_1 \) and all \( \hat{z}_1 \in \hat{E}_1 \) and such that \( \sum B_n \) is a weakly unconditionally Cauchy (WUC) series in the space \( L(E_1) \),

(iii) there is a sequence \( \{A_n\} \subseteq K(E) \) of \( w(E_1, E_1) \)-continuous operators such that, for all \( z \in E \), \( A_n(z) \to z \) in the \( w(E, E_1) \)-topology,

(iv) \( A : X \to E \) is \( w(X, \hat{X}) \)-\( w(E, E_1) \) continuous and bounded,

(v) \( B : E \to Y \) is \( w(E, E_1) \)-\( w(Y, \hat{Y}) \) continuous and bounded,

(vi) the unit balls of the spaces \( X \) and \( \hat{Y}_1 \) are compact in the \( w(X, \hat{X}) \) and \( w(\hat{Y}_1, Y_1) \) topologies, respectively.

Then certain convex blocks of \( \{IBA_iA\} \) are (WUC) and, in each point \( x \in X \), they converge to \( IT(x) \) in the \( w(Y_1, \hat{Y}_1) \) topology.

Moreover, if the operator \( T \) is not compact then the sequence space \( c_0 \) is isomorphically contained in \( \overline{\text{span}} \{BA_iA\} \subset K(X, Y) \).

**Remark 1.** As we shall see below the condition (i) is usually automatically verified by considering \( Y \) embedded into an injective superspace \( Y_1 \). A version of the Proposition where \( X \) is a quotient of an \( l_1(\Gamma) \) is also possible.

**Remark 2.** Note that the following condition (ii)' implies the conditions (ii) in the Propositions 1 and 1a.

(ii)' There are \( w(E_1, E_1) \)-continuous operators \( B_n \in K(E_1) \) for all \( n \in N \), such that \( \sum_{n=1}^{\infty} B_n(e_1) = e_1 \) where the countable sum converges unconditionally in the norm for all \( e_1 \in E_1 \).

Moreover, if \( \sum B_n(e_1) \) converges unconditionally to \( e_1 \) for all \( e_1 \in E_1 \) then (ii)' together with the other assumptions of the Propositions also imply that certain
convex blocks of \( \{ BA_i A \} \) are, for each point \( x \in X \), unconditionally converging to \( T(x) \). This applies also in the Corollaries 1-4 and in the Theorem 1. Indeed, the set \( \{ \sum_{n=1}^{m} \pm B_n(e_1); m \in N \} \) is bounded for all \( e_1 \in E_1 \). The uniform boundedness principle then yields that the set \( \{ \| \sum_{n=1}^{m} \pm B_n \|; m \in N \} \) is bounded again.

**Corollary 1 ([4])**. Let \( T \in L_{w^*}(X^*, Y) \) be a noncompact operator. Suppose that \( Y \) has the compact approximation property and that \( Y \) is a subspace of a separable Banach space \( Y_1 \) such that \( Y_1 \) has the unconditional compact approximation property. Then \( c_0 \subset K_{w^*}(X^*, Y) \).

**Proof**. It is enough to choose \( E = Y \) and \( B = \text{Id}_Y \) in the Proposition. \( \square \)

Similarly we get the more general and new

**Corollary 2.** Let \( T = BA : X^* \rightarrow Y \) be a factorization of a noncompact operator \( T \) through a Banach space \( E \) such that \( A : X^* \rightarrow E \) is \( w^*-w \) continuous and \( B \in L(E, Y) \). Suppose that \( E \) has the compact approximation property and that \( E \) is a subspace of a separable Banach space \( E_1 \) such that \( E_1 \) has an unconditional compact approximation property. Then \( c_0 \subset K_{w^*}(X^*, Y) \).

**Proof**. We choose in the Proposition for \( Y_1 \) any injective Banach space containing \( Y \), e.g. \( l_\infty(B_Y) \), \( \overset{\rightharpoonup}{Y}_1 = Y_1^* \) and \( \overset{\rightharpoonup}{Y} = Y^* \).

Similar statement may be given e.g. for the case when \( A \) is \( w^*-w^* \) continuous and \( B \) is \( w^*-w \) continuous.

**Corollary 3.** Let \( T = BA : X^* \rightarrow Y \) be a factorization of a noncompact operator \( T \) through a Banach space \( E^* \) such that \( A : X^* \rightarrow E^* \) is \( w^*-w^* \) continuous and \( B \in L(E^*, Y) \) is \( w^*-w \) continuous. Suppose that \( E \) has the compact approximation property and that \( E \) is a quotientspace of a separable Banach space \( E_1 \) such that the imbedding \( J : E^* \rightarrow E_1^* \) is \( w^*-w^* \) continuous and such that \( E_1 \) has an unconditional compact approximation property. Suppose further that \( I \) is an imbedding of the space \( Y \) into the Banach space \( Y_1 \) such that the operator \( B \) has an extension to a \( w^*-w \) continuos operator \( \tilde{B}_1 : E_1^* \rightarrow Y_1 \) in the sense that \( \tilde{B}J(e) = IB(e) \) for all \( e \in E^* \). Then \( c_0 \subset K_{w^*}(X, Y) \).

The next theorem is in fact a consequence of our Proposition 1. Because it has a less technical formulation, we prefer to state it separately.

**Theorem 1.** Let \( T \in L(X, Y) \) be a noncompact operator and let \( T = BA \) be a factorization of \( T \) through a Banach space \( E \). Suppose that
either

(1) $E$ is isomorphic to a quotient space of a Banach space $E_1$, the space $E^*$ has the compact approximation property and the space $E_1$ has the shrinking unconditional compact approximation property

or

(2) $E$ is isomorphic to a subspace of a Banach space $E_1$, the space $E^{**}$ has the compact approximation property and the space $E_1^*$ has the shrinking unconditional compact approximation property.

Then the sequence space $c_0$ is isomorphically contained in $K(X,Y)$.

Proof. Case 1. We shall apply the Proposition to the noncompact operator $T^* = A^*B^*: Y^* \to X^*$. Let $Q: E_1 \to E$ be the surjection operator. It is well known that we may choose a linear surjection $q: l_1(\Gamma) \to X$. The lifting property of $l_1(\Gamma)$ yields an operator $S: l_1(\Gamma) \to E_1$ such that $Aq = QS$. In the Proposition we may now substitute for the space $Y$ the space $X^*$, for the isomorphic embedding $J: E \to E_1$ the $w^*-w^*$ continuous embedding $Q^*: E^* \to E_1^*$, for the isomorphic embedding $I: Y \to Y_1$ the $w^*-w^*$ continuous embedding $q^*: X^* \to l_\infty(\Gamma)$, for $B$ the mapping $S^*$. Further we substitute $l_1(\Gamma)^{**}$ for $E_1$, $X$ for $\hat{Y}$ and $Y$ for $\hat{X}$. Then (i)-(iv) are easily seen to be satisfied. The condition (vi) means in our case that the closed unit balls $B_{Y_1}$ and $B_{X^{**}}$ are $w^*$-compact. To check (iv) it is sufficient to observe that the operators $q^*A^*A_i^*B^*$ are $w^*-w$ continuous. But this follows immediately because these operators are $w^*-w^*$ continuous and compact. Similarly we observe that (v) holds. Proposition 1 now gives that $c_0 \subset \overline{\text{span}}\{A^*A_i^*B^*\}$ which means that $c_0 \subset \overline{\text{span}}\{BA_iA\}$.

Case 2. In this case $E^*$ is isomorphic to a quotient of the space $E_1^*$ and we may apply the case (1) to the noncompact operator $T^* = A^*B^*: Y^* \to X^*$. □

Remark 4. The case (2) in the above Theorem may also be obtained by applying the Proposition directly to the factorization of $T^{**}: X^{**} \to Y^{**}$ through the space $E^{**}$ where $E^{**} \subset E_1^{**}$. We also embed $Y$ into an injective Banach space.

Remark 5. If in the Theorem 1 the operator $A: X \to E$ is weakly compact or if $l_1 \not\subset E^*$ then the assumption concerning $E$ in (1) may be that only $E$ has the compact approximation property and in (2) that only $E^*$ has the compact approximation property. Indeed, first we notice that we may assume that $A^*$ is unconditionally convergent (otherwise $A^*$ would fix a copy of $c_0$ and thus $c_0 \subset X^*$ and this in turn would imply that $c_0 \subset K(X,Y)$). If now $l_1 \not\subset E^*$ then by Pelczynski (see [10]) $E^*$ has the property (V) and thus $A^*$ is weakly compact.

The last consequence of the previous results is a slight generalization of a result due to Feder
Corollary 4 ([5]). Let $X$ be isomorphic to a factor space of a Banach space $X_1$, $X_1$ having the shrinking unconditional compact approximation property. Let the space $X^*$ have the compact approximation property and let $L(X,Y)$ contain a noncompact operator $M$. Then $c_0 \subset K(X,Y)$ isomorphically.

Proof. We apply Theorem 1 (1) (after taking $X = E$, $X_1 = E_1$) to the operator $T = M \text{Id}_X$; it then yields a copy of $c_0$ inside $\text{span}\{MA_i\} \subset K(Z,Y)$. □

References


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